

# Moduli of curves and multiple roots

Emre Can Sertöz

## Abstract

We compactify the moduli space of curves and multiple square roots of a line bundle. The obvious compactification behaves badly, thus we provide a compactification using “synchronized” torsion-free sheaves. The resulting moduli space is smooth and the parametrized objects have a good moduli interpretation.

## 1 Introduction

Let  $C$  be a proper smooth curve with a line bundle  $N$ . Consider an  $m$ -tuple of square roots of  $N$ , that is, a sequence of line bundles  $L_1, \dots, L_m$  such that  $L_i^{\otimes 2} \simeq N$ .

Given a degeneration of  $(C, N)$  to a singular stable curve, it is known how each root  $L_i$  will deform. However, if these individual degenerations are not “synchronized” then we do not get a satisfactory theory for the degeneration of  $m$ -tuples of roots. More precisely, the associated moduli stack is non-normal and the underlying degenerations are unnatural.

In this article, we describe how to synchronize these degenerations and prove that the resulting moduli problem is represented by a smooth Deligne-Mumford stack. Moreover, we show that the resulting degenerations are geometrically meaningful.

In the paper [FJR13] a compactification of the moduli of various arrangements of roots of the canonical bundle is constructed using line bundles on twisted curves. We adopt the more geometric point of view and use line bundles on quasi-stable curves, or equivalently torsion-free sheaves on stable curves, to compactify our moduli space.

These results are applied in the author’s thesis to the study of configurations of theta hyperplanes via degeneration. We wish to discuss these results in an upcoming paper [Ser17].

### 1.1 Statement of the result for multiple spin curves

Here we present a special case of our main result when applied to tuples of spin structures on curves. Since the language and notation for spin curves are well established, we can state our result quickly. Again for the sake of familiarity, and only for this section, let us work over the field of complex numbers  $\mathbb{C}$ .

Take  $N = \omega_C$  to be the sheaf of differentials. A pair  $(L, \alpha: L^{\otimes 2} \xrightarrow{\sim} \omega_C)$ , with  $L$  a line bundle, is called a spin structure on  $C$  and the triplet  $(C, L, \alpha)$  is called a spin curve. The moduli space of smooth genus  $g$  spin curves is denoted by  $\mathcal{S}_g$ . The forgetful functor  $\mathcal{S}_g \rightarrow \mathcal{M}_g$  to the moduli space of smooth genus  $g$  curves induces a finite map between the coarse moduli spaces.

There is a compactification  $\overline{\mathcal{S}}_g \rightarrow \overline{\mathcal{M}}_g$  of  $\mathcal{S}_g$  over the moduli space of stable curves, whose coarse moduli scheme over  $\mathbb{C}$  was originally constructed by Cornalba [Cor89] and later the fine moduli stack was constructed in greater generality by Jarvis [Jar98].

The initial goal of this paper has been to find a “good” compactification for the  $m$ -fold product  $\mathcal{S}_g^{\times m} = \mathcal{S}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{S}_g$ , i.e., the moduli space of curves with an  $m$ -tuple of spin structures. We achieved this goal in greater generality, not restricting ourselves to the roots of the canonical bundle. For now however, we will continue to describe our main result for this specific case.

Let us point out that the obvious compactification  $\overline{\mathcal{S}}_g^{\times m} = \overline{\mathcal{S}}_g \times_{\overline{\mathcal{M}}_g} \cdots \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g$  is non-normal (see Appendix B). There is another problem with this compactification: the objects it parametrizes are unnatural as we will see below.

The moduli space  $\overline{\mathcal{S}}_g$  parametrizes limit spin curves. These are triplets  $(X, L, \alpha: L^{\otimes 2} \rightarrow \omega_X)$  where  $X$  is a *quasi-stable* curve (Definition A.2),  $L$  is a line bundle on  $X$  and  $\alpha$  is almost an isomorphism (Definition A.7). The forgetful map  $\overline{\mathcal{S}}_g \rightarrow \overline{\mathcal{M}}_g$  sends  $(X, L, \alpha)$  to the stabilization  $C$  of  $X$ .

When we consider the product  $\overline{\mathcal{S}}_g^{\times m}$ , the objects we parametrize would then be  $m$ -tuples of the form  $(\pi_i: X_i \rightarrow C, L_i, \alpha_i)_{i=1}^m$  where each  $\pi_i$  is the stabilization map. In other words, the stabilizations are identified but not the quasi-stable curves  $X_i$ . So that we end up with  $m$  line bundles on  $m$  different curves!

A good compactification of  $\mathcal{S}_g^{\times m}$  should parametrize objects that are of the form  $(X, \{L_i, \alpha_i\}_{i=1}^m)$  where  $X$  is quasi-stable and each  $(X, L_i, \alpha_i)$  is a limit spin curve, possibly after a partial stabilization of  $X$ .

If we leave it at that, our moduli space would not have finite fibers over  $\overline{\mathcal{M}}_g$ . To overcome this problem, we require that for each  $i, j$  the line bundles  $L_i^{\otimes 2}$  and  $L_j^{\otimes 2}$  are isomorphic around the unstable components on which they have the same degree (see Definition A.19).

Let us denote the resulting moduli space by  $\overline{\mathcal{S}}_g^m$ . Then our main result becomes:

**Theorem 1.1.** *The moduli space  $\overline{\mathcal{S}}_g^m$  is proper and the inclusion  $\mathcal{S}_g^{\times m} \hookrightarrow \overline{\mathcal{S}}_g^m$  is dense and open. The forgetful map  $\overline{\mathcal{S}}_g^m \rightarrow \overline{\mathcal{M}}_g$  induces a finite map over the coarse moduli spaces. Furthermore, the stack  $\overline{\mathcal{S}}_g^m$  is smooth.*

The objects parametrized by  $\overline{\mathcal{S}}_g^m$  are built to be used for enumerative problems. Therefore the last condition, giving us the smoothness of  $\overline{\mathcal{S}}_g^m$ , is particularly valuable.

To phrase our main result precisely and in appropriate generality, we have to introduce quite a bit of technical machinery. So we begin our introduction once again, this time with the language of torsion-free sheaves which we will use throughout the paper.

## 1.2 Conventions

As we are going to work with  $m$ -tuples of roots, fix once and for all an integer  $m \geq 1$ .

**Definition 1.2.** Let  $\mathcal{M}$  be an algebraic stack. Then, a *stable curve* over  $\mathcal{M}$  is a proper, flat, finitely presented morphism  $\pi: \mathcal{C} \rightarrow \mathcal{M}$  whose geometric fibers are reduced, connected and of dimension 1, with at worst nodal singularities and such that the relative dualizing sheaf  $\omega_{\mathcal{C}/\mathcal{M}}$  is relatively ample.

It is well known that the moduli space of line bundles on a nodal curve is not proper. One way to compactify this space is via torsion-free sheaves of rank-1.

We do not need this result at the moment but this fact motivates the following definition.

**Definition 1.3** (Jarvis). A *torsion-free sheaf on a stable curve*  $\mathcal{C} \rightarrow \mathcal{M}$  is a coherent  $\mathcal{O}_{\mathcal{C}}$ -module  $\mathcal{E}$  which is flat and of finite presentation over  $\mathcal{M}$  such that over each  $s \in \mathcal{M}$  the fiber  $\mathcal{E}|_{\mathcal{C}_s}$  has no associated primes of height one.

An elegant definition of a square root of a line bundle is provided by the following:

**Definition 1.4** (Deligne, Jarvis). Let  $\mathcal{E}$  be a rank-1 torsion-free sheaf on a curve  $\mathcal{C} \rightarrow \mathcal{M}$  and  $\mathcal{N}$  a line bundle on  $\mathcal{C}$ . Let  $\delta: \mathcal{E} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{E}^\vee$  be an isomorphism. Then the pair  $(\mathcal{E}, \delta)$  will be called a *(square) root of  $\mathcal{N}$* .

**Definition 1.5.** Given a coherent module  $\mathcal{E}$  and a line bundle  $\mathcal{N}$  on a scheme  $\mathcal{X}$ , a homomorphism  $b: \mathcal{E}^{\otimes 2} \rightarrow \mathcal{N}$  will be called a *bilinear form*, with  $\mathcal{N}$  understood from context.

Notice that a bilinear form induces two maps  $b^l, b^r: \mathcal{E} \rightarrow \mathcal{E}^\vee \otimes \mathcal{N}$  where  $\mathcal{E}^\vee = \text{hom}(\mathcal{E}, \mathcal{O}_{\mathcal{X}})$ ,  $b^r(e) = b(e, \_)$  and  $b^l(e) = b(\_, e)$ .

**Definition 1.6.** Given a bilinear form  $b: \mathcal{E}^{\otimes 2} \rightarrow \mathcal{N}$ , if both  $b^r$  and  $b^l$  are isomorphisms then  $b$  is said to be *non-degenerate*. If  $b^r = b^l$  then  $b$  is *symmetric*, and then  $b$  factors through the symmetrizing map  $\mathcal{E}^{\otimes 2} \rightarrow \text{Sym}^2 \mathcal{E}$ .

**Notation for symmetric powers.** We will adopt an unusual notational custom and for any  $A$ -module  $E$  write the  $d$ -th symmetric product  $\text{Sym}_A^d(E)$  simply as  $E^d$ , and given  $\mu: E \rightarrow F$  we will denote by  $\mu^d$  the induced map  $E^d \rightarrow F^d$ . The same goes for sheaves of modules and morphisms between them. In compensation, we will write out tensor powers and direct sums explicitly as  $E^{\otimes d}$  and  $E^{\oplus d}$ , respectively.

We will now state our working definition of a root, which is equivalent to the definition of Deligne and Jarvis above.

**Definition 1.7.** Let  $\mathcal{E}$  be a rank-1 torsion-free sheaf on a curve  $\mathcal{C} \rightarrow \mathcal{M}$  and  $\mathcal{N}$  a line bundle on  $\mathcal{C}$ . Let  $b: \mathcal{E}^2 \rightarrow \mathcal{N}$  be a non-degenerate symmetric form. Then the pair  $(\mathcal{E}, b)$  will be called a *(square) root of  $\mathcal{N}$  on  $\mathcal{C}/\mathcal{M}$* .

**Remark 1.8.** To see that Definition 1.7 is equivalent to Definition 1.4 proceed as follows. Given  $\delta: \mathcal{E} \xrightarrow{\sim} \mathcal{N} \otimes \mathcal{E}^\vee$  we obtain a non-degenerate bilinear form  $b: \mathcal{E}^{\otimes 2} \rightarrow \mathcal{N}$ . We will prove in Remark 2.8 that any such  $b$  is in fact symmetric, giving a non-degenerate  $b: \mathcal{E}^2 \rightarrow \mathcal{N}$ . For the converse, given  $(\mathcal{E}, b)$  let  $\delta := b^r = b^l$ .

**Remark 1.9.** Jarvis in [Jar98] works with non-degenerate forms  $E^{\otimes 2} \rightarrow \mathcal{N}$ , although they are automatically symmetric. This is no problem when working with a single root. In considering tuples of roots however, carrying around the kernel of  $E^{\otimes 2} \rightarrow E^2$  is disruptive and so we consider the equivalent formulation of roots using symmetric powers.

**Remark 1.10.** If we let  $V \hookrightarrow \mathcal{C}$  denote the open locus on which  $\mathcal{E}$  is free, then the smooth locus of the map  $\mathcal{C} \rightarrow \mathcal{M}$  is contained in  $V$ . In addition,  $b$  is an isomorphism on  $V$ .

**Definition 1.11.** An isomorphism  $\mu: (\mathcal{E}, b) \rightarrow (\mathcal{E}', b')$  of roots is defined to be an isomorphism of the underlying sheaf of modules  $\mu: \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$  such that  $b = b' \circ \mu^2$ .

**Notation.** By a DM stack we will mean a Deligne–Mumford stack.

### 1.3 Setting up the problem

Fix an excellent base scheme  $S$  defined over  $\mathbb{Z}[1/2]$  (e.g.,  $S = \mathbb{Z}[1/2]$  or  $S = \mathbb{C}$  will do) and let  $\mathcal{M} \rightarrow S$  be a DM stack, locally of finite type over  $S$ .

Fix a stable curve  $\mathcal{C} \rightarrow \mathcal{M}$  of genus  $g \geq 2$ , which need not be generically smooth. In addition, fix a line bundle  $\mathcal{N}$  on  $\mathcal{C}$  having absolutely bounded degree (see Definition 1.13). This is a very weak condition, see Remark 1.14.

This ensures that twisting the line bundle  $\mathcal{N}$  by a sufficiently high power of a relatively ample bundle, such as  $\omega_{\mathcal{C}/\mathcal{M}}$ , will kill relative cohomology and  $\mathcal{N}$  will, upon twisting, be relatively base point free.

Given any  $T \rightarrow \mathcal{M}$  we can pullback the curve  $\mathcal{C}$  and  $\mathcal{N}$  to get a stable curve  $\mathcal{C}_T \rightarrow T$  together with a line bundle  $\mathcal{N}_T$ .

**Definition 1.12.** Let  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  be the category fibered in groupoids whose objects over  $T \rightarrow \mathcal{M}$  are roots of  $\mathcal{N}_T$ . Similarly, let  $\mathcal{S}(\mathcal{N}) \subset \overline{\mathcal{S}}(\mathcal{N})$  be the subcategory consisting of roots that are locally free.

The arguments in [Jar98] imply that  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is an algebraic space (see Section 5.1.1), which compactify  $\mathcal{S}(\mathcal{N}) \rightarrow \mathcal{M}$ . In fact, [Jar98] deals in a slightly more restricted setting where  $\mathcal{C} \rightarrow \mathcal{M}$  is the universal curve over the moduli space of stable curves of genus  $g \geq 2$ . We replaced this with the boundedness condition on  $\mathcal{N}$ , which seems to be the key in establishing the algebraicity of  $\overline{\mathcal{S}}(\mathcal{N})$ .

Moreover, we will show in Section 5.1 that the arguments presented *loc.cit.* imply that  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow S$  is a Deligne–Mumford. Moreover, we show that  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow S$  is smooth if  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  is smooth.

Denote by  $\mathcal{S}^m(\mathcal{N})$  the  $m$ -fold product  $\mathcal{S}(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{S}(\mathcal{N})$ . Our goal is to find a “good” compactification of  $\mathcal{S}^m(\mathcal{N})$ . Something which  $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  fails to do, as it is non-normal (see Appendix B) and the objects parametrized by this fiber product are not geometrically meaningful (see Proposition A.16).

The reader interested only in spin curves may simply take  $\mathcal{M}$  to be the moduli space of stable curves of genus  $g$ ,  $\mathcal{C} \rightarrow \mathcal{M}$  to be the universal curve over it and  $\mathcal{N}$  to be the relative dualizing sheaf  $\omega_{\mathcal{C}/\mathcal{M}}$ .

#### 1.3.1 Roots of higher degree and twisted curves

In literature,  $r$ -th roots of line bundles have already been studied: from the perspective of torsion-free sheaves in [Jar98] and from the equivalent perspective of quasi-stable curves in [CCC07]. We will only consider  $m$ -tuples of *square* roots ( $r = 2$ ) because in passing to  $r \geq 3$  a hefty technical price has to be paid even in defining the roots. We avoided this because we feel the theory of twisted curves are better suited to handle the theory of roots when  $r \geq 3$ .

As we mentioned in the beginning, a compactification of such tuples (of  $r$ -th roots of the canonical bundle, and its variations) is already constructed in [FJR13] using line bundles on twisted curves.

Nevertheless, the definition of *square* roots in terms of torsion-free sheaves is much shorter and far more accessible for geometric problems than twisted curves. With that said, we hope our current pursuit is well justified.

### 1.3.2 Absolutely bounded degree

**Definition 1.13.** If there exists a constant  $c \in \mathbb{Z}$  such that on any component  $Y$  of any geometric fiber of  $\mathcal{C} \rightarrow \mathcal{M}$  we have  $\deg \mathcal{N}|_Y \geq c$  then  $\mathcal{N}$  will be said to have *absolutely bounded degree*.

**Remark 1.14.** This boundedness condition is weak enough that unless  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  has geometric fibers with infinitely many connected components, the condition is automatically satisfied. In any case, if  $\mathcal{N} = \omega_{\mathcal{C}/\mathcal{M}}^{\otimes l}$  for any  $l \in \mathbb{Z}$ , then  $\mathcal{N}$  has absolutely bounded degree (see Sublemma 4.1.10 [Jar98] for  $l = 1$ , the idea readily generalizes to all  $l \in \mathbb{Z}$ ).

### 1.3.3 A remark about Artin's approximation theorem

Some of the results cited throughout this paper were written when Artin's approximation theorem was known to be applicable only over a restricted class of excellent rings. Since then this restriction has been lifted (see [CJ02]) and we will freely use the cited results over arbitrary excellent rings.

## 1.4 Statement of the result

We will define  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{M}$  in Section 5.2. Summarizing Theorem 5.29, Corollary 5.47 and Section 5.4 we get:

**Theorem 1.15.**  $\overline{\mathcal{S}}^m(\mathcal{N})$  is a DM stack, locally of finite type, proper and quasi-finite over  $\mathcal{M}$ .

With further assumptions on  $\mathcal{M}$  we can also say more about  $\overline{\mathcal{S}}^m(\mathcal{N})$ . The most useful ones are given by Theorem 5.48 and Corollary 5.50. These are:

**Theorem 1.16.** If  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  is smooth, then so is  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow S$ .

**Theorem 1.17.** If  $\mathcal{C} \rightarrow \mathcal{M}$  is generically smooth, then  $\mathcal{S}^m(\mathcal{N}) \hookrightarrow \overline{\mathcal{S}}^m(\mathcal{N})$  is a dense open immersion.

Possibly the most studied setting is when  $\mathcal{M} = \overline{\mathcal{M}}_{g,n}$  and when  $\mathcal{C} = \overline{\mathcal{C}}_{g,n} \rightarrow \mathcal{M}$  is the universal curve. Denote by  $\sigma_1, \dots, \sigma_n: \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{C}}_{g,n}$  the  $n$  markings. The hypothesis of the theorems above are satisfied and we have the following corollary.

**Corollary 1.18.** For any  $a_1, \dots, a_n \in \mathbb{Z}$  let  $\mathcal{N} = \mathcal{O}_{\overline{\mathcal{C}}_{g,n}}(\sum_{i=1}^n a_i \sigma_i)$  or  $\mathcal{N} = \omega_{\overline{\mathcal{C}}_{g,n}/\overline{\mathcal{M}}_{g,n}}(\sum_{i=1}^n a_i \sigma_i)$ . Then  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow S$  is a proper smooth Deligne–Mumford stack over  $\mathcal{M}$  and  $\mathcal{S}^m(\mathcal{N}) \hookrightarrow \overline{\mathcal{S}}^m(\mathcal{N})$  is a dense open immersion.

This implies in particular that the coarse moduli space of  $\overline{\mathcal{S}}^m(\mathcal{N})$  exists, is finite over the coarse moduli of  $\overline{\mathcal{M}}_{g,n}$  and is projective over  $S$  (see Proposition 5.55).

This corollary agrees with the results of [FJR13] when  $m \geq 2$  and with [Jar98] when  $m = 1$ .

**Remark 1.19.** These results may be more interesting for some when phrased in the language of limit roots and quasi-stable curves. For this reason in Appendix A we make the equivalence between limit roots and (torsion-free) roots explicit.

## 1.5 Overview

In Section 2 we concentrate on the formal neighbourhood of the node of a curve over an algebraically closed field and describe how a root degenerates together with the node. This section is largely expository and is included for easy reference of technical lemmas.

In Section 3 we define how to “synchronize” the deformation of a sequence of  $m$  roots at a node. We then study the deformation of synchronized roots together with the node.

In Section 4 we bring together the results of the past two sections to study the deformations of a curve together with an  $m$ -tuple of synchronized roots. This section provides us with the local description of  $\overline{\mathcal{S}}^m(\mathcal{N})$ .

Finally, in Section 5 we define  $\overline{\mathcal{S}}^m(\mathcal{N})$  in full generality and then prove that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is a DM stack. We end the section by establishing various properties of  $\overline{\mathcal{S}}^m(\mathcal{N})$  such as being smooth and proper over  $S$  provided that  $\mathcal{M}$  is reasonably nice.

In Appendix A we give another, more geometric, interpretation of synchronized  $m$ -tuples of roots in terms of line bundles on blow-ups of curves in the same vein as [Cor89] and [CCC07].

In Appendix B we prove that the product  $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  will, in general, be non-normal.

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## 2 Universal deformation of a node with a root

In this section we define the deformation of a node together with a root of a line bundle and then give the universal deformation. This amounts to bringing together the results available in the literature, i.e., in [Fal96] and [Jar98].

Faltings’ paper studies torsion-free sheaves of finite rank, also with a non-degenerate quadratic form. Jarvis’ paper studies rank-1 torsion-free sheaves as  $r$ -th roots of line bundles. However, rank-1 torsion-free sheaves considered as a square root (i.e.,  $r = 2$ ) lies in the intersection of these two papers and are by

far the simplest to consider. Therefore, a treatment of this special case is quite revealing.

In addition, we provide a more detailed proof of Proposition 5.4.3 in [Jar98] for square roots. We package this result in Theorem 2.30.

## 2.1 Conventions

In this section and the next we will be concerned about (infinitesimal) deformations of an affine scheme, as these are always affine it is convenient to work in the dual category of algebras instead of schemes. However, the arrows are mostly written so that when we apply Spec to the diagrams they look familiar.

### 2.1.1 Notation

- $k$  is an algebraically closed field of characteristic  $\neq 2$ .
- $\Lambda$  is a complete noetherian local ring with residue field  $k$ .
- $\text{Art}_\Lambda$  is the category of Artinian local  $\Lambda$ -algebras with residue field  $k$ .
- $\hat{\text{Art}}_\Lambda$  is the category of complete noetherian local  $\Lambda$ -algebras  $(R, \mathfrak{m}_R)$  such that for each  $n \geq 1$  we have  $R/\mathfrak{m}_R^n \in \text{Art}_\Lambda$ .

## 2.2 Deformations of a node

**Definition 2.1.** Let  $\bar{A} := k[[x, y]]/(xy) \leftarrow k$ . We will refer to  $\bar{A}$  as the *standard node*. By a *deformation of the node (over  $R$ )* we will refer to tuples  $(A \leftarrow R, \iota)$  where  $R \in \hat{\text{Art}}_\Lambda$ ,  $A$  is a complete local flat  $R$ -algebra and  $\bar{A} \xleftarrow{\iota} A$  fits into a *Cartesian* diagram:

$$\begin{array}{ccc} \bar{A} & \xleftarrow{\iota} & A \\ \uparrow & & \uparrow \\ k & \longleftarrow & R \end{array}$$

where the map  $k \leftarrow R$  is the residue map. Isomorphism of deformations are defined in the usual way.

**Definition 2.2.** The *functor of deformations of the node* is a functor  $G: \text{Art}_\Lambda \rightarrow (\text{Sets})$  which maps  $R$  to the set of isomorphism classes of deformations of the node over  $R$ .

The following theorem is folklore. The proof follows essentially the same steps as in [Stacks, Tag 0CBX].

**Theorem 2.3.** *The deformation  $(\Lambda[[x, y, t]]/(xy - t) \leftarrow \Lambda[[t]], j: t \mapsto 0)$  is universal, i.e.,  $\Lambda[[\tau]]$  pro-represents  $G$ . In particular, given any deformation  $(A \leftarrow R, \iota) \in G(R)$  we have a unique map  $\Lambda[[t]] \rightarrow R: t \mapsto \pi \in \mathfrak{m}_R$  which induces an isomorphism  $A \simeq R[[x, y]]/(xy - \pi)$ .*

**Remark 2.4.** We can define a functor  $\mathfrak{m}: \text{Art}_\Lambda \rightarrow (\text{Sets}): R \mapsto \mathfrak{m}_R$  by attaching the maximal ideal to a local ring. Another way to interpret Theorem 2.3 is to say that  $G$  and  $\mathfrak{m}$  are naturally isomorphic. More precisely, the natural transformation  $G \rightarrow \mathfrak{m}$  can be defined as  $(R[[x, y]]/(xy - \pi) \leftarrow R, \iota) \in G(R) \mapsto (\pi \in \mathfrak{m}_R)$ .

## 2.3 Deformations of a root

**Remark 2.5.** Any line bundle on a curve restricted to the complete local ring of one of its node will be (non-canonically) isomorphic to the trivial line bundle. For this reason, we will study the roots of the trivial line bundle on a deformation of the node.

**Set-up 2.6.** Throughout this subsection let  $(A \leftarrow R, \iota)$  be a deformation of the node and let  $E$  be an  $R$ -flat and  $R$ -relatively torsion-free rank-1  $A$ -module, *which is not free*.

**Remark 2.7.** We exclude the case where  $E$  is free simply because its deformation theory is trivial. However, free roots play a role in later chapters.

To define the notion of a root we need to discuss bilinear forms momentarily.

**Remark 2.8.** With  $E$  as in Set-up 2.6, if  $b: E^{\otimes 2} \rightarrow A$  is a bilinear form then  $b$  is symmetric. Indeed, since  $E$  is rank-1 the map  $E^{\otimes 2} \rightarrow \text{Sym}^2 E$  is generically an isomorphism, with the kernel being  $(x, y)$ -torsion. Since  $A$  has no  $(x, y)$ -torsion,  $b$  kills this kernel and factors through  $\text{Sym}^2 E$ .

**Definition 2.9.** A tuple  $(E, b)$  with  $E$  as in Set-up 2.6 and with  $b: \text{Sym}^2 E \rightarrow A$  a non-degenerate bilinear form on  $E$  will be called a *root*. An isomorphism between two roots  $(E, b)$  and  $(E', b')$  is an isomorphism  $\mu: E \rightarrow E'$  such that  $b' \circ \mu^2 = b$ . We will denote  $b' \circ \mu^2$  by  $\mu^* b'$ .

Although we are excluding the case where  $E$  is free, we will often want to refer to this case. Hence we will also introduce the following terminology.

**Definition 2.10.** A tuple  $(E, b: E^2 \xrightarrow{\sim} A)$  where  $E$  is a *free* rank-1  $A$ -module will be referred to as a *free root*. We say  $(E, b)$  is a *possibly free root* if  $(E, b)$  is allowed to be either a free root or a (non-free) root.

Let  $(\bar{E}, \bar{b})$  be a root on the standard node and  $(E, b)$  a root on  $(A \leftarrow R, \iota)$ . We will write  $\iota_* E$  for  $E \otimes_A \bar{A}$  and  $\iota_* b$  for the map  $\iota_* E^2 \rightarrow \bar{A}$  induced from  $b$ .

**Definition 2.11.** Let  $j: \iota_* E \xrightarrow{\sim} \bar{E}$  be an isomorphism such that  $\bar{b} \circ j^2 = \iota_* b$ . Then we will refer to the tuple  $(E, b, j)$  as a *deformation of the root*  $(\bar{E}, \bar{b})$ . The map  $j$  will be called a *restriction map*. An isomorphism of deformations is an isomorphism of roots commuting with the restriction maps.

## 2.4 Standard roots

Let  $R \in \hat{\text{Art}}_\Lambda$  and  $A = R[[x, y]]/(xy - \pi)$  for some  $\pi \in \mathfrak{m}_R$ . Define  $\iota: A \rightarrow \bar{A} = k[[x, y]]/(xy)$  using  $R \rightarrow R/\mathfrak{m}_R = k$ .

### 2.4.1 Faltings' construction

Let  $p, q \in R$  be such that  $pq = \pi$ . Define  $2 \times 2$  matrices with entries in  $A$ :

$$\alpha = \begin{pmatrix} x & p \\ q & y \end{pmatrix}, \quad \beta = \begin{pmatrix} y & -p \\ -q & x \end{pmatrix}$$

Clearly  $\alpha\beta = \beta\alpha = 0$  but moreover we get an exact infinite periodic complex (see [Fal96]):

$$\dots \rightarrow A^{\oplus 2} \xrightarrow{\alpha} A^{\oplus 2} \xrightarrow{\beta} A^{\oplus 2} \xrightarrow{\alpha} A^{\oplus 2} \xrightarrow{\beta} A^{\oplus 2} \rightarrow \dots$$



**Definition 2.12.** Define  $E(p, q) \subset A^{\oplus 2}$  to be the image of  $\alpha$  or, equivalently, the kernel of  $\beta$ . Truncating the complex above we get a free resolution of  $E(p, q)$ , whenever we refer to the standard resolution of  $E(p, q)$  this is the one we mean.

**Remark 2.13.** It is straight forward to check that  $E(p, q)$  is relatively torsion-free. Moreover, with some more work, one can see that  $E$  is  $R$ -flat, see Construction 3.2 of [Fal96].

If  $p$  or  $q$  is invertible, then  $E(p, q)$  is free. As we are not dealing with free roots, from now on we assume  $p, q \in \mathfrak{m}_R$ . Note that this implies  $\pi \in \mathfrak{m}_R^2$ .

It is easy to see that the dual  $E(p, q)^\vee = \text{hom}(E(p, q), A)$  is naturally isomorphic to  $E(q, p)$ . In particular, when  $p = q$  the module  $E(p, p)$  is self-dual.

**Definition 2.14.** The natural pairing gives us a map  $s: E(p, p)^2 \rightarrow A$  which we will call the *standard map*. For  $\bar{E} = E(0, 0)$  on  $\bar{A}$  denote the standard map by  $\bar{s}$ .

**Remark 2.15.** Theorem 2.20 below states that any root on  $(A \leftarrow R, \iota)$  is isomorphic to  $(E(p, p), s)$  for some  $p \in \mathfrak{m}_R$ . In particular, we have a root iff  $\pi \in \mathfrak{m}_R^2$ .

**Definition 2.16.** Let us refer to  $(E(p, p), s)$  as a *standard root on  $(A \leftarrow R, \iota)$* .

**Remark 2.17.** There may be different values of  $p$  which give non-isomorphic roots. But on  $\bar{A} \leftarrow k$  there is only one standard root. As an example take  $R = k[t]/(t^2)$  and  $A = R[[x, y]]/(xy)$ . Then  $E(t, t)$  and  $E(0, 0)$  are not isomorphic even as modules (one could apply Proposition 3.3 of [Fal96] to see this).

#### 2.4.2 Properties of standard roots

Given any  $b$  on  $E(p, q)$  we can lift it to  $\text{Sym}^2 A^{\oplus 2} \twoheadrightarrow \text{Sym}^2 E(p, q)$  to get a morphism  $\tilde{b}: \text{Sym}^2 A^{\oplus 2} \rightarrow A$ . Letting  $e_1, e_2$  be the standard generators of  $A^{\oplus 2}$  and  $e_1^2, e_1 e_2, e_2^2$  the corresponding generators of  $\text{Sym}^2 A^{\oplus 2}$  we may uniquely identify  $b$  with the values  $b_0 := \tilde{b}(e_1^2), b_1 := \tilde{b}(e_1 e_2), b_2 := \tilde{b}(e_2^2)$ . By abuse of notation we will write  $b = (b_0, b_1, b_2)$ .

**Lemma 2.18.** For a standard root  $(E(p, p), s)$  we have  $s = (x, p, y)$ .

*Proof.* Let  $\langle \cdot, \cdot \rangle$  denote the natural pairing  $A^{\oplus 2} \times A^{\oplus 2} \rightarrow A$ . The identification of  $E(p, p)^\vee$  with  $E(p, p)$  makes it clear that if  $e, f \in E(p, p)$  and  $u, v \in A^{\oplus 2}$  are such that  $e = \alpha(u)$  and  $f = \alpha(v)$  then we have

$$s(e, f) = \langle u, \alpha(v) \rangle = \langle \alpha(u), v \rangle.$$

Now, direct computation yields the result. □

**Lemma 2.19.** Any root  $(E(p, p), b)$  on  $A \leftarrow R$  is isomorphic to  $(E(p, p), s)$ .

*Proof.* Lemma 5.4.10 [Jar98] states that  $b = (ax, b_1, awy)$  where  $a \in A^*$  and  $w \in R^*$  such that  $wp = p$ . Note here that as we are working with square roots of line bundles, the hypothesis of the cited lemma is satisfied (as stated in Corollary 5.4.9 *loc.cit.*).

Let  $v$  be a square root of  $w$  and consider the isomorphism  $\mu: E \rightarrow E$  which descends from multiplication by  $\begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$  on  $A^{\oplus 2}$ . Clearly  $\mu^* b = a(vb_0, b_1, v^{-1}b_2) =$

$va(x, v^{-1}b_1, y)$ . By scaling  $E$  we may now assume  $va = 1$  and  $b = (x, b_1, y)$  where we changed  $b_1$ .

Since  $\alpha(y, 0) = \alpha(0, p)$  and  $\alpha(0, x) = \alpha(p, 0)$  we see that  $pb_2 = yb_1$  and  $pb_0 = xb_1$ . Which means  $y(b_1 - p) = x(b_1 - p) = 0$  (we used  $wp = p$ ). But  $\text{Ann}_A(x, y) = 0$  hence  $b_1 = p$ .  $\square$

**Theorem 2.20** (Faltings). *Let  $(E, b)$  be a root on  $A$ . Then  $\exists p \in \mathfrak{m}_R$  such that  $(E, b) \xrightarrow{\sim} (E(p, p), s)$ .*

*Proof.* We are going to apply Theorem 3.7 in [Fal96] to torsion-free sheaves of rank-1. In fact, Faltings classifies non-degenerate quadratic forms on  $E$  whereas we have non-degenerate bilinear forms  $b: E^2 \rightarrow A$  which is the same.

Faltings' Theorem implies that  $(E, b) \simeq (E(p, p), b')$  for some  $p \in \mathfrak{m}_R$  and  $b'$ . But now we can apply Lemma 2.19 to deduce the desired result.  $\square$

We now wish to describe isomorphisms of roots. Since we know that all roots are isomorphic to  $(E(p, p), s)$  for some  $p \in \mathfrak{m}_R$  with  $p^2 = \pi$ , it suffices to calculate  $\text{Iso}((E(p, p), s), (E(q, q), s))$  for  $p, q \in \mathfrak{m}_R$  such that  $p^2 = q^2 = \pi$ .

Let  $e_1, e_2 \in A^{\oplus 2}$  be the *standard basis* and let  $\xi_1, \xi_2 \in E(p, p)$  be the images of  $e_1$  and  $e_2$  respectively. Let us refer to  $\xi_1, \xi_2$  as the *standard generators* of  $E(p, p)$ . Note that any automorphism of  $E(p, p)$  can be lifted to a map  $A^{\oplus 2} \rightarrow A^{\oplus 2}$ .

**Notation 2.21.** If  $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}: A^{\oplus 2} \rightarrow A^{\oplus 2}$  descends to  $\mu \in \text{hom}(E(p, p), E(q, q))$  then we will write  $\mu = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ .

**Lemma 2.22.** *We have:*

$$\text{Iso}((E(p, p), s), (E(q, q), s)) = \left\{ \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix} \mid \varepsilon_1, \varepsilon_2 \in \{\pm 1\}, q = \varepsilon_1 \varepsilon_2 p \right\}$$

*Proof.* This is proven in a similar way to Proposition 4.1.12 of [Jar98], so we will give a sketch. An easy observation is that we can choose a lift of  $\mu$  of the form

$$\begin{pmatrix} u_+(x) & v_+(x) \\ v_-(y) & u_-(y) \end{pmatrix}$$

where  $u_+, v_+ \in R[[x]] \subset A$  and  $u_-, v_- \in R[[y]] \subset A$ . Now we simply have to calculate what it means to have  $\mu^* s = s$  in terms of  $u_{\pm}, v_{\pm}$ . Using that  $x$  (resp.  $y$ ) does not annihilate  $R[[x]]$  (resp.  $R[[y]]$ ) we see immediately that  $v_{\pm} = 0$ ,  $u_{\pm} \in \{\pm 1\}$  is forced. Then  $q = u_+ u_- p$ .  $\square$

**Definition 2.23.** Let  $\mu: E(p, q) \xrightarrow{\sim} E(p', q')$  be an isomorphism. Notice that the free resolutions attached to these modules canonically identify the central fibers with  $k^{\oplus 2}$ . Denote the restriction of  $\mu$  to the central fibers by  $\mu(0): k^{\oplus 2} \rightarrow k^{\oplus 2}$ .

**Remark 2.24.** Suppose  $\mu: (E(p, p), s) \xrightarrow{\sim} (E(q, q), s)$ . Then restricting  $\mu^2$  to the central fibers gives us  $\mu^2(0): (k^{\oplus 2})^2 \rightarrow (k^{\oplus 2})^2$ . It is immediate to check that  $\mu^2(0) = \text{Id}$  iff  $\mu(0) = \pm \text{Id}$  and  $\mu^2(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \neq \text{Id}$  iff  $\mu(0) = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . In other words, when  $p \neq 0$  then  $\mu^2(0) = \text{Id}$  iff  $p = q$ .

**Definition 2.25.** On  $(A \leftarrow R, \iota)$  there is a *natural restriction map* from  $E(p, q)$  to  $\bar{E} = E(0, 0)$  which is the map  $r$  completing the diagram below:

$$\begin{array}{ccc} A^{\oplus 2} & \xrightarrow{\iota} & \bar{A}^{\oplus 2} \\ \alpha(p, q) \downarrow & & \downarrow \alpha(0, 0) \\ E(p, q) & \xrightarrow{r} & \bar{E} \end{array}$$

**Remark 2.26.** The triplet  $(E(p, p), s, r)$  is a deformation of the root  $(\bar{E}, \bar{s})$ .

**Remark 2.27.** Lemma 2.22 implies that choosing a restriction map  $r$  rigidifies the root. That is,  $\text{Aut}(E(p, p), s, r) = 1$ . The following result takes this observation one step further.

**Proposition 2.28.** *Suppose that  $(E, b, j)$  is a deformation of  $(\bar{E}, \bar{s})$ . Then there exists precisely one  $p \in R$  such that  $(E, b, j) \simeq (E(p, p), s, r)$ . Moreover, this isomorphism is unique.*

*Proof.* Uniqueness of the isomorphism follows from Remark 2.27. By Theorem 2.20 we know that  $(E, b) \simeq (E(p, p), s)$  for some  $p \in R$ . Picking one such isomorphism we may assume  $(E, b, j) = (E(p, p), s, j)$  for some  $j$ . However, with our choice of identification,  $j$  is not necessarily equal to the natural restriction  $r$ .

Let  $\gamma = j \circ r^{-1}: (\bar{E}, \bar{s}) \rightarrow (\bar{E}, \bar{s})$ . Then  $\gamma$  is uniquely defined by  $\gamma(0) \in \{\pm \text{Id}, \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$ . An isomorphism  $\mu: (E(p, p), s) \xrightarrow{\sim} (E(q, q), s)$  commutes with  $j$  and  $r$  iff  $\iota_* \mu: (\bar{E}, \bar{s}) \xrightarrow{\sim} (\bar{E}, \bar{s})$  is the inverse of  $\gamma$ . Having classified such  $\mu$  in Lemma 2.22 we know that there exists precisely one  $q$  and one  $\mu$  which will restrict to  $\gamma^{-1}$ .  $\square$

## 2.5 The universal deformation

**Definition 2.29.** Call the tuple  $(\bar{A} \leftarrow k, \bar{E}, \bar{s})$  the *rooted node*. A deformation of the node together with a deformation of the root, which looks like  $(A \leftarrow R, i, E, b, j)$ , will be called a *deformation of the rooted node*.

Let the functor  $F: \text{Art}_\Lambda \rightarrow (\text{Sets})$  associate to  $R$  the isomorphism classes of deformations of the rooted node.

**Theorem 2.30.** *The ring  $\Lambda[[\tau]]$  pro-represents  $F$ . The universal family is given by  $(\Lambda[[x, y, \tau]](xy - \tau^2) \leftarrow \Lambda[[\tau]], \tau \mapsto 0, E(\tau, \tau), s, r)$ .*

*Proof.* Given any deformation of the rooted node  $(A \leftarrow R, \iota, E, b, j)$  we wish to show that there exists a unique map  $\varphi: \Lambda[[\tau]] \rightarrow R$  such that  $A$  is canonically isomorphic to  $\Lambda[[x, y, \tau]](xy - \tau^2) \otimes_{\Lambda[[\tau]]} R$  and  $\varphi_*(E(\tau, \tau), s, r) \simeq (E, b, j)$ . Furthermore, that this isomorphism is unique.

Proposition 2.28 shows that there exists a *unique*  $p \in R$  such that  $(E, b, j)$  is (uniquely) isomorphic to  $(E(p, p), s, r)$ , moreover this implies  $A = R[[x, y]]/(xy - \pi)$  with  $\pi = p^2$ . Define  $\varphi$  by  $\tau \mapsto p$ . Since the maps  $s$  and  $r$  are natural, the pushforward of  $(E(\tau, \tau), s, r)$  is (uniquely) isomorphic to  $(E(p, p), s, r)$ .

Choosing any other map  $\tau \mapsto q$  would give a root that is not isomorphic to  $(E, b, j)$ . Thus we have proven the existence and uniqueness of the map  $\varphi$  of the desired form.  $\square$

**Remark 2.31.** As in Remark 2.4, this theorem allows us to identify the functor  $F$  with the functor  $\mathbf{m}: R \mapsto \mathbf{m}_R$ . This time the identification is achieved by mapping  $(R[[x, y]]/(xy - p^2), \iota, E(p, p), s, r) \in F(R)$  to  $p \in \mathbf{m}_R$ . If we identify the functor of deformations of the node  $G$  with  $\mathbf{m}$  as in Remark 2.4 then the forgetful functor  $F \rightarrow G$  corresponds to the squaring map  $\mathbf{m} \rightarrow \mathbf{m}: (p \in \mathbf{m}_R) \mapsto (p^2 \in \mathbf{m}_R)$ .

### 3 Universal deformation of a node with multiple roots

Having fixed a positive integer  $m$ , we will suppress it from notation when referencing  $m$ -tuples of roots.

**Definition 3.1.** Let  $(A \leftarrow R, \iota)$  be a deformation of the node. A *multiple root* is a tuple  $(\mathcal{R}, \Phi)$  where  $\mathcal{R} = (E_i, b_i)_{i=1}^m$  is a sequence of (non-trivial) roots and  $\Phi = (h_i: E_1^2 \xrightarrow{\sim} E_i^2)$  is a sequence of isomorphisms with  $h_1 := \text{Id}$  and the rest satisfying the following conditions:

- $\exists u_i: E_1 \xrightarrow{\sim} E_i$  such that  $u_i^2 = h_i$
- $b_i \circ h_i = b_1$ .

The data  $\Phi$  will be called a *synchronization* on the sequence of roots  $\mathcal{R}$ . An *isomorphism of multiple roots* is a sequence of isomorphisms between the roots commuting with the synchronizations.

**Remark 3.2.** If  $E_1$  is isomorphic to  $E(0, 0)$  then so is  $E_i$  for all  $i$ . Then,  $A$  must be the trivial deformation of the node over  $R$ . Unless this is the case, it follows from Remark 2.24 that the synchronization  $\Phi$  is uniquely determined from the sequence of roots  $\mathcal{R}$ .

**Remark 3.3.** In other words, the definition above is symmetric and does not even require an ordering of the index set. Indeed, given a multiple root  $(\mathcal{R}, \Phi)$  and any pair of indices  $i, j \in \{1, \dots, m\}$  then we can define  $h_{ij} := h_j \circ h_i^{-1}: E_i^2 \xrightarrow{\sim} E_j^2$  which will satisfy the two conditions above. Conversely, if we are given isomorphisms  $(h_{ij})_{i,j}$  with  $h_{ii} = \text{Id}$ ,  $h_{ij}$  the square of an isomorphism and  $b_i = b_j \circ h_{ij}$  then  $(h_i := h_{1i})$  gives us a synchronization.

Although we will stick to the definition above for the next two sections, we will eventually need to consider not just sequences of roots but sequences of possibly free roots. Hence we define the following:

**Definition 3.4.** Let  $\mathcal{R}$  be a sequence of possibly free roots. Let  $\Phi$  be a synchronization on the subsequence  $\mathcal{R}'$  of  $\mathcal{R}$  consisting of non-free roots. If  $\mathcal{R}' \neq \emptyset$  then  $(\mathcal{R}, \Phi)$  will be called a *generalized multiple root* and when  $\mathcal{R}' = \emptyset$  then  $\mathcal{R}$  itself will be called a *generalized multiple root*. As before  $\Phi$  is called a *synchronization*.

#### 3.1 Deformations of multiple roots

Let  $(A \leftarrow R, \iota)$  be a deformation of the standard node  $\bar{A} \leftarrow k$ . Let  $(\bar{\mathcal{R}}, \bar{\Phi})$  be a multiple root on the standard node.

**Definition 3.5.** A *deformation of  $(\bar{\mathcal{R}}, \bar{\Phi})$  on  $A$*  is a tuple  $(\mathcal{R}, \Phi, j)$  where  $(\mathcal{R}, \Phi)$  is a multiple root and  $j$  is a sequence of restriction maps, i.e., a sequence of isomorphisms  $j: \iota_* \mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$ . Moreover, we ask that  $j$  commute with the synchronization  $\Phi$  and  $\bar{\Phi}$ .

### 3.1.1 Conventions

- On the standard node  $\bar{A} \leftarrow k$  denote the  $i$ -th root by  $(\bar{E}_i, \bar{s}_i)$  and let each of these be the standard root, i.e.,  $(\bar{E}_i, \bar{s}_i) = (\bar{E}, \bar{s})$ . As described in Remark 2.24, we have two options for each of the maps  $\bar{h}_i: \bar{E}_1^2 \rightarrow \bar{E}_i^2$ . Either  $h_i = \text{Id}$  or  $h_i \neq \text{Id}$ . Fix a sequence  $\bar{\Phi} = (\bar{h}_i)_{i=1}^m$ . From now on  $(\bar{\mathcal{R}}, \bar{\Phi})$  will denote this multiple root.
- Define a sequence  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$  where  $\varepsilon_1 = 1$ ,  $\varepsilon_i = 1$  if  $\bar{h}_i = \text{Id}$  and  $\varepsilon_i = -1$  if  $\bar{h}_i \neq \text{Id}$ . Clearly we can recover  $\bar{\Phi}$  from  $\varepsilon$ .
- Given a deformation  $(\mathcal{R}, \Phi, j)$  use the following letters for the underlying objects:  $\mathcal{R} = (E_i, b_i)_{i=1}^m$ ,  $\Phi = (h_i)_{i=1}^m$  and  $j = (j_i)_{i=1}^m$ .
- In the rest of this section, equality between deformations of roots is used to designate the unique isomorphism between them.

### 3.1.2 The universal deformation

**Lemma 3.6.** *Let  $(\mathcal{R}, \Phi, j)$  be a deformation of  $(\bar{\mathcal{R}}, \bar{\Phi})$  on  $(A \leftarrow R, \iota)$ . Then  $\exists! p \in R$  such that for all  $i$  we have  $(E_i, b_i, j_i) = (E(\varepsilon_i p, \varepsilon_i p), s, r)$ .*

*Proof.* By Proposition 2.28 we know that  $\forall i \exists! p_i \in R$  such that  $(E_i, b_i, j_i) = (E(p_i, p_i), s, r)$ . Let  $p = p_1$ . Since the existence of  $\Phi$  forces all roots  $(E_i, b_i)$  to be isomorphic we may apply Lemma 2.22 to conclude  $p_i \in \{\pm p\}$ .

If  $p = 0$  then there is nothing more to prove so assume  $p \neq 0$ . Then the sign of  $p_i$  is completely determined by  $h(0)$  by Remark 2.24. However, the restriction maps  $r$  identify all the central fibers of  $E_i$  and  $\bar{E}_i$  so that  $h(0) = \text{Id}$  iff  $\bar{h}(0) = \text{Id}$ . Thus  $p_i = p$  iff  $\varepsilon_i = 1$ .  $\square$

Define  $H: \text{Art}_\Lambda \rightarrow (\text{Sets})$  to be the functor associating to each  $R$  the set of isomorphism classes of deformations of  $(\bar{\mathcal{R}}, \bar{\Phi})$ .

**Theorem 3.7.** *The functor  $H$  is pro-represented by  $\Lambda[[\tau]]$  with the universal deformation given by  $(\Lambda[[\tau, x, y]]/(xy - \tau^2) \leftarrow \Lambda[[\tau]], \tau \mapsto 0, (E(\varepsilon_i \tau, \varepsilon_i \tau), s, r)_{i=1}^m)$ .*

**Remark 3.8.** We omitted the synchronizations from the description of the family because by Remark 3.2 the synchronizations are uniquely defined given these roots.

*Proof.* Let  $(A \leftarrow R, \iota)$  be a deformation of the node and let  $(\mathcal{R}, j) = (E_i, b_i, j_i)_{i=1}^m$  together with the synchronizations  $\Phi = \{h_i\}_{i=1}^m$  be a deformation of  $(\bar{\mathcal{R}}, \bar{\Phi})$ . Any map  $\varphi: \Lambda[[\tau]] \rightarrow R$  is uniquely defined by the choice of  $p \in R$  for which  $\tau \mapsto p$ . Lemma 3.6 tells us that there is a unique  $p \in R$  for which  $\varphi_*(E(\varepsilon_i \tau, \varepsilon_i \tau), s, r) = (E_i, b_i, j_i)$ . This proves the existence and uniqueness of  $\varphi$  provided we show that the synchronizations agree.

This is done by reducing to the fiber over the node as in Lemma 3.6 at which point compatibility of the synchronizations is immediate.  $\square$

We will be interested in generalized multiple roots and their deformations. Here is a key lemma which implies that the deformation functor of a multiple root does not change if we add free roots. See Remark 3.11 for a precise statement.

This time let  $R \in \hat{\text{Art}}_\Lambda$  and  $(A \leftarrow R, \iota)$  a deformation of the standard node. Note that  $A$  is complete with respect to the ideal  $\mathfrak{m}_R \cdot A$ . We sketch a proof of the following elementary lemma.

**Lemma 3.9.** *Let  $(\bar{L}, \bar{b}: \bar{L}^{\otimes 2} \xrightarrow{\sim} \bar{A})$  be a free root on  $\bar{A} \leftarrow k$  and let  $(A \leftarrow R)$  be a deformation of  $\bar{A} \leftarrow k$  (which is now allowed to be smooth). Then there is a deformation, unique up to a unique isomorphism, of the free root  $(\bar{L}, \bar{b})$  on  $(X/R, \iota)$ .*

*Proof.* Since  $A$  is complete with respect to  $\mathfrak{m}_R \cdot A$  we just have to show that there exists a unique lift from  $R/\mathfrak{m}_R^n$  to  $R/\mathfrak{m}_R^{n+1}$ . Existence is clear. What has to be shown is that there exists a unique isomorphism between any two lifts. But this can be reduced to showing that successive lifts of square roots of invertible elements are unique, which is true.  $\square$

**Remark 3.10.** The proof of this lemma works in greater generality, and we need the more general statement as well. If we have a deformation of a nodal curve then in Zariski neighbourhoods of smooth points, the deformation is trivial. In particular, with the base complete local, the argument above works verbatim. Since roots are always free on the smooth locus of the fibers, we can lift any root uniquely along the smooth part of a deformation.

**Remark 3.11.** Lemma 3.9 has the following useful consequence. Let  $\bar{\mathcal{R}}$  be a generalized root and  $\bar{\mathcal{R}}'$  the multiple root obtained from  $\bar{\mathcal{R}}$  by removing the free roots. Then, Lemma 3.9 implies that the deformation functors of  $\bar{\mathcal{R}}$  and  $\bar{\mathcal{R}}'$  are identified by forgetting the free roots. Here, if  $\bar{\mathcal{R}}' = \emptyset$  then by a deformation of  $\bar{\mathcal{R}}'$  we will mean the deformation of the underlying node.

## 3.2 Further comments on our definition of multiple roots

Instead of working with a sequence of roots, we chose to work with

- (a) a sequence of *isomorphic* roots,
- (b) together with the square of an isomorphism between each pair of roots.

Suppose we dropped these 2 conditions and defined the functor  $F_m$  of  $m$ -tuples of roots without any restrictions. Then clearly  $F_m$  is the  $m$ -fold product of  $F$  (deformation of roots) over  $G$  (deformation of nodes). As both of these functors are pro-representable, so is  $F_m$ . For instance  $F_2 = F \times_G F$  is pro-represented by  $\Lambda[[\tau]] \times_{\Lambda[[\tau^2]]} \Lambda[[\tau]] \simeq \Lambda[[\tau_1, \tau_2]]/(\tau_1^2 - \tau_2^2)$ , which is non-normal.

What if we include the first condition and drop the second condition? Define the functor  $F'_m$  of  $m$ -tuples of *isomorphic* roots.

**Claim 3.12.** *The functor  $F'_m$  is not representable.*

*Proof.* For convenience let  $m = 2$  and  $F' := F'_2$ , though the proof works for all  $m > 1$ . Recalling that the functor of deformation of roots,  $F$ , is identified with  $R \mapsto \mathfrak{m}_R$  it becomes clear that  $F': R \mapsto \{(p, \varepsilon p) \mid p \in \mathfrak{m}_R, \varepsilon = \pm 1\}$ .

To show that the functor  $F'$  is not representable we check Schlessinger's first condition (denoted by  $H_1$  in [Sch68]). Let  $k[\epsilon_i] \simeq k[t]/(t^2)$  and  $k[\epsilon_1, \epsilon_2] = k[\epsilon_1] \times_k k[\epsilon_2]$ . Then the map  $F'(k[\epsilon_1, \epsilon_2]) \rightarrow F'(k[\epsilon_1]) \times F'(k[\epsilon_2])$  is given by  $(a\epsilon_1 + b\epsilon_2, \nu) \mapsto (a\epsilon_1, \nu) \times (b\epsilon_2, \nu)$  where  $\nu \in \{\pm 1\}$ . Since the pair of signs has to be equal on the image (both equal to  $\nu$ ), this map can not be surjective. Thus  $H_1$  is violated and  $F'$  can not be representable.  $\square$

On the other hand, if one were to pick a sequence of roots together with isomorphisms between them (as opposed to the square of the isomorphisms) then one simply recovers the functor  $F_1$ . This defeats the purpose, because pairs of free roots are locally isomorphic but since we only identify the squares of these line bundles, we can only determine a local isomorphism up to sign. Thus,  $F_1$  does not adequately generalize the moduli problem of having  $m$  locally free roots on smooth curves.

## 4 Universal deformation of stable curves with multiple roots

### 4.1 Universal deformation of a stable curve

Let  $X/k$  be a stable curve of genus  $g$  with  $n$  nodes  $x_1, \dots, x_n \in X$ .

**Set-up 4.1.** For each  $i$ , let  $\hat{\mathcal{O}}_i := \hat{\mathcal{O}}_{X, x_i} \simeq k[[x, y]]/(xy)$ . Let  $G_i: \text{Art}_k \rightarrow (\text{Sets})$  be the functor of deformations of  $\hat{\mathcal{O}}_i \leftarrow k$ . By Theorem 2.3 we can pick a formal variable  $t_i$  and identify  $G_i$  with  $\text{hom}(\Lambda[[t_i]], \_)$ .

**Definition 4.2.** For  $R \in \hat{\text{Art}}_k$ , a curve  $\mathcal{X}/R$  together with a  $k$ -isomorphism  $\iota: X \xrightarrow{\sim} \mathcal{X}|_k$  is said to be a *deformation of  $X$  over  $R$* . Pullbacks and isomorphisms of deformations are defined in the usual way.

**Remark 4.3.** When  $R$  is artinian, the underlying topological spaces of a deformation  $\mathcal{X}/R$  of  $X$  and  $X$  itself are naturally identified. We will use this identification between the points of  $X$  and points of  $\mathcal{X}$  without further remark.

**Definition 4.4.** The functor  $D_X: \text{Art}_k \rightarrow (\text{Sets})$  assigning to each  $R$  the set of isomorphism classes of deformations of  $X$  over  $R$  is called the *functor of infinitesimal deformations of  $X$* .

Given a deformation  $(\mathcal{X}/R, \iota)$  of  $X$ , the tuple  $(\hat{\mathcal{O}}_{\mathcal{X}, x_i} \leftarrow R, \iota)$  is a deformation of  $\hat{\mathcal{O}}_i \leftarrow k$ . Thus, for each of the nodes  $x_i \in X$  we get a natural transformation  $D_X \rightarrow G_i$ .

**Theorem 4.5** (Deligne-Mumford [DM69]). *Let  $T = \Lambda[[t_1, \dots, t_{3g-3}]]$ . There exists a universal deformation  $(\mathcal{C}/T, u)$  of  $X$  over  $T$ , through which  $T$  pro-represents  $D_X$ . Moreover, for each  $i = 1, \dots, n$ , the map  $D_X \rightarrow G_i$  corresponds to the map  $\Lambda[[t_i]] \rightarrow T: t_i \mapsto t_i$ .*

**Remark 4.6.** This theorem accounts for the first  $n$  generators of  $T$  in the sense that  $t_i$  parametrizes the deformation of the node  $x_i$ . Although the order of the other variables are arbitrarily chosen here they could be grouped together into meaningful chunks. Let  $W = \langle t_{n+1}, \dots, t_{3g-3} \rangle$  be the  $k$ -vector space generated by the remaining parameters. If  $Y \rightarrow X$  is the normalization of any irreducible component of  $X$  and  $y_1, \dots, y_r \in Y$  are the points mapping to nodes of  $X$ , then the universal deformation of the marked curve  $(Y, y_1, \dots, y_r)$  accounts for a  $3g(Y) - 3 + r$  dimensional subspace of  $W$ . When  $g(Y) \geq 2$  then  $r$  of these dimensions can be identified with the deformation of the markings on the “constant” curve. In this way we can further partition  $W$  into meaningful subspaces.

#### 4.1.1 Deforming the line bundle

We need to pause for a minute and consider our situation before we can proceed further. Although we studied the universal deformation of the curve  $X$  as well as the universal deformation of each of its nodes, with and without roots, we can not make direct use of them. This is because an arbitrary line bundle on  $X$  may not extend to the universal deformation of  $X$ .

We deal with this problem by specifying a global framework in which we specify how the line bundle we are interested in deforms. This is the reason for our set-up in Section 1.3, which we recall again.

We have a genus  $g$  stable curve  $\mathcal{C} \rightarrow \mathcal{M}$  over a Deligne–Mumford stack  $\mathcal{M}$  of finite type over an excellent scheme  $S \rightarrow \mathbb{Z}[\frac{1}{2}]$ . In addition, we specified a line bundle  $\mathcal{N}$  on  $\mathcal{C}$ . Now, by a curve  $X/k$  together with a line bundle  $\tilde{\mathcal{N}}$  we really mean a morphism  $\text{Spec } k \rightarrow \mathcal{M}$ . Pulling back  $\mathcal{C} \rightarrow \mathcal{M}$  and  $\mathcal{N}$  to  $\text{Spec } k$  gives us  $(X/k, \tilde{\mathcal{N}})$ .

Note however that the pair  $(X, \tilde{\mathcal{N}})$  does not determine the map  $\text{Spec } k \rightarrow \mathcal{M}$ , since  $\mathcal{M}$  need not be universal in anyway. So we really need to specify the morphism  $\text{Spec } k \rightarrow \mathcal{M}$  and not just the tuple  $(X, \tilde{\mathcal{N}})$ . Nevertheless, for readability, we may often refer to a morphism  $\text{Spec } k \rightarrow \mathcal{M}$  simply as  $(X, \tilde{\mathcal{N}})$ . Here is one example of how we will abuse notation.

**Definition 4.7.** Let  $k$  be any field and  $R$  a complete local ring with residue field  $k$ . Let  $(X, \tilde{\mathcal{N}})$  be obtained from  $p: \text{Spec } k \rightarrow \mathcal{M}$ . By a *deformation of  $(X, \tilde{\mathcal{N}})$  on  $R$*  (but really a deformation of  $p$  on  $R$ ) we will mean a tuple  $(P, \iota)$  such that the following diagram 2-commutes:

$$\begin{array}{ccc} \text{Spec } k & \xrightarrow{p} & \mathcal{M} \\ \downarrow & \searrow \iota & \\ \text{Spec } R & \xrightarrow{P} & \mathcal{M} \end{array}$$

where the vertical arrows is the quotient map  $R \twoheadrightarrow k$ .

**Remark 4.8.** If  $P$  gives us  $(X/R, \mathcal{N})$  then  $\iota$  is an isomorphism  $(X|_k, \mathcal{N}|_k) \xrightarrow{\sim} (X, \tilde{\mathcal{N}})$  over  $k$ . For this reason, in accordance with our previous notation, we will denote deformations of  $(X, \tilde{\mathcal{N}})$  as tuples  $(X/R, \mathcal{N}, \iota)$ .

#### 4.1.2 Geometric formal neighbourhoods

**Definition 4.9.** Let  $k$  be any field. Then by Cohen structure theorem there exists a universal coefficient ring, which we will denote by  $\mathfrak{o}_k$ . So that any complete local ring with residue field  $k$  contains a copy of  $\mathfrak{o}_k$ . If  $\text{char } k = 0$  then  $\mathfrak{o}_k = k$ .

Let  $s = \text{Spec } k \in S$  be any point. The complete local ring  $\hat{\mathcal{O}}_{S,s}$  pro-represents the functor  $\text{Art}_{\mathfrak{o}_k} \rightarrow (\text{Sets})$  defined by  $A \mapsto \text{hom}_s(A, S)$ , where the subscript  $s$  indicates that the morphisms must restrict to  $s$  on the residue field.

For any morphism  $s = \text{Spec } k \rightarrow S$ , with  $k$  any field we can still define a functor  $Q_s: \text{Art}_{\mathfrak{o}_k} \rightarrow (\text{Sets})$  via the rule  $A \mapsto \text{hom}_s(A, S)$ . If  $s \rightarrow S$  factors through the point  $s' = \text{Spec } k' \in S$  then  $Q_s$  is pro-represented by the complete local ring  $\hat{\mathcal{O}}_{S,s'} \otimes_{\mathfrak{o}_{k'}} \mathfrak{o}_k$ .

**Definition 4.10.** For any point  $s = \text{Spec } k \rightarrow S$  let  $\hat{\mathcal{O}}_{S,s}$  denote the complete local ring pro-representing the functor  $Q_s$  above.



For the Deligne–Mumford stack  $\mathcal{M}$  and a point  $p = \operatorname{Spec} k \rightarrow \mathcal{M}$  the functor  $Q_p$  can be defined just like  $Q_s$ . This functor is seen to be pro-representable by using any étale chart.

**Definition 4.11.** For any point  $p = \operatorname{Spec} k \rightarrow \mathcal{M}$  the complete local ring pro-representing  $Q_p$  will be denoted by  $\hat{\mathcal{O}}_{\mathcal{M},p}$ .

Let  $s = \operatorname{Spec} k \rightarrow S$  be a geometric point and  $p: \operatorname{Spec} k \rightarrow \mathcal{M}$  be a point of finite type lying above  $s$ . Let  $\Lambda = \hat{\mathcal{O}}_{S,s}$  and let  $\tilde{\Lambda} := \hat{\mathcal{O}}_{\mathcal{M},p}$ . Denote by  $X/k$  the fiber of  $\mathcal{C} \rightarrow \mathcal{M}$  over  $p$  and let  $\tilde{N}$  be the restriction of  $N$  to  $X$ .

By Theorem 4.5 the universal deformation functor  $D_X$  of  $X$  can be represented by the ring  $\Lambda[[t_1, \dots, t_{3g-3}]]$ . For convenience let us refer to  $\operatorname{hom}_{\hat{\mathcal{O}}_{S,s}}(\hat{\mathcal{O}}_{\mathcal{M},p}, \_)$  as  $D_{\mathcal{M},p}$ . The natural map  $D_{\mathcal{M},p} \rightarrow D_X$  corresponds to a map:

$$\Lambda[[t_1, \dots, t_{3g-3}]] \rightarrow \tilde{\Lambda}.$$

We will study how introducing multiple roots changes this local behavior. To do this, let us first define deformations over  $\mathcal{M}$ .

## 4.2 Roots on nodal curves

Let  $\tilde{N}$  be a line bundle on the stable curve  $X/k$ . Let  $(\mathcal{X}/R, \mathcal{N}, \iota)$  be a deformation of  $(X, \tilde{N})$ .

**Definition 4.12.** For each node  $x_v \in X$  denote by  $U_{x_v}(\mathcal{X}) := \operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X},x_v} \rightarrow \mathcal{X}$  the *formal neighbourhood* of  $x_v$  in  $\mathcal{X}$ . The pullback map to  $U_{x_v}(\mathcal{X})$  will be denoted by  $\hat{x}_v^*$ .

**Remark 4.13.** Given a root  $(\mathcal{E}, b)$  of  $\mathcal{N}$  on  $\mathcal{X}$ , the pullback  $\hat{x}_v^*(\mathcal{E}, b)$  is a (possibly free) root on  $U_v(\mathcal{X})$ , in the sense of Definition 2.10 (keeping in mind Remark 2.5). The codomain of  $\hat{x}_v^*b$  is no longer canonically isomorphic to the structure sheaf but to  $\hat{x}_v^*\mathcal{N}$ . This makes no difference for the theory in the sense that the deformation functors are isomorphic.

## 4.3 Multiple roots on nodal curves

As we move onto sequences of roots, let us recall our fixed integer  $m \geq 1$ . Suppose we have a sequence  $\mathcal{R} := (\mathcal{E}_i, b_i)_{i=1}^m$  of roots on  $\mathcal{X}$ . Then  $\hat{x}_v^*\mathcal{R}$  is a sequence of (possibly free) roots. We want to define a synchronization for such sequences.

**Definition 4.14.** For a root  $(\mathcal{E}, b)$  on  $\mathcal{X}$ , by *singularities of  $\mathcal{E}$*  we will mean the subset of the nodes on which  $\mathcal{E}$  is not-free.

Re-index the nodes if necessary so that we get an  $n' \leq n$  such that  $x_v$  is a singularity of one of the roots if and only if  $v \leq n'$ .

**Definition 4.15.** Let  $\mathcal{R}_{x_v} := \hat{x}_v^*\mathcal{R}$  be the sequence of possibly free roots on  $U_v(\mathcal{X})$ . A synchronization, as in Definition 3.4, on  $\mathcal{R}_{x_v}$  will be denoted by  $\Phi_{x_v}$ .

**Definition 4.16.** Let  $\mathcal{R}$  be a sequence of roots on  $\mathcal{X}$ . For each  $v \leq n'$  let  $\Phi_{x_v}$  be a synchronization on  $\mathcal{R}_{x_v}$ . Let  $\Phi = (\Phi_{x_v})_{v=1}^{n'}$ . Then the pair  $(\mathcal{R}, \Phi)$  will be called a *multiple root (of  $\mathcal{N}$ )*.

**Remark 4.17.** If for some  $v$  the sequence  $\mathcal{R}_{x_v}$  contains non-isomorphic roots, then there exists no synchronization  $\Phi_{x_v}$ . In this case,  $\mathcal{R}$  simply can not be made into a multiple root.

## 4.4 Deformation theory

Fix a multiple root  $(\bar{\mathcal{R}}, \bar{\Phi})$  on  $X/k$ . Let  $\xi = (X/k, \bar{\mathcal{R}}, \bar{\Phi})$ .

**Definition 4.18.** If  $(\mathcal{R}, \Phi)$  is a root on the deformation  $(\mathcal{X}/R, \iota)$  and  $j: \iota^*\mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$  is a sequence of isomorphisms then we will call  $j$  a *restriction map*. If  $j$  commutes with the synchronization  $\Phi$  and  $\bar{\Phi}$  then  $(\mathcal{R}, \Phi, j)$  will be called a *deformation of  $(\bar{\mathcal{R}}, \bar{\Phi})$* , and  $(\mathcal{X}/R, \iota, \mathcal{R}, \Phi, j)$  is called a *deformation of  $(X/k, \bar{\mathcal{R}}, \bar{\Phi})$* .

**Set-up 4.19.** As before, order the nodes so that there is an  $n' \leq n$  such that  $\bar{\mathcal{R}}$  has non-free roots at  $x_v$  iff  $v \leq n'$ . Then, for any deformation  $(\mathcal{R}, \Phi, j)$  of  $(\bar{\mathcal{R}}, \bar{\Phi})$  one of the roots in  $\hat{x}_v^*\mathcal{R}$  is non-free iff  $v \leq n'$ .

**Definition 4.20.** The functor  $D_\xi: \text{Art}_\Lambda \rightarrow (\text{Sets})$  which takes  $R$  to the set of isomorphism classes of deformations of  $\xi$  is called the *functor of infinitesimal deformations of  $\xi$* .

Let  $(\mathcal{X}/R, \iota)$  be a deformation of  $X/k$  where we allow  $R \in \hat{\text{Art}}_\Lambda$ . Denote  $\bar{\mathcal{R}}$  as  $(\bar{\mathcal{C}}_i, \bar{b}_i)_{i=1}^m$ .

**Lemma 4.21.** *Suppose that for each  $1 \leq v \leq n'$  we are given a deformation  $\mathfrak{R}_v = (\mathcal{R}_v, \Phi_v, j_v)$  of  $\hat{x}_v^*(\bar{\mathcal{R}}, \bar{\Phi})$  on  $U_v(\mathcal{X})$ . Then there exists a unique deformation  $(\mathcal{R}, \Phi, j)$  of  $\mathfrak{R} = (\bar{\mathcal{R}}, \bar{\Phi})$  on  $\mathcal{X}$  which pulls back to  $(\mathcal{R}_v, \Phi_v, j_v)$  on each  $U_v(\mathcal{X})$ .*

*Proof.* We use Grothendieck Existence Theorem to reduce the question to the formal neighbourhood of the central fiber. Thus we may assume  $R$  is artinian.

Let  $\mathfrak{m}_R$  be the maximal ideal of  $R$ . Let  $R_l := R/\mathfrak{m}_R^l$  and  $X_l := \mathcal{X}|_{R_l}$  for all  $l \geq 0$ . For each  $l$  and  $v \leq n'$  we can pullback  $\mathfrak{R}_v$  to the formal neighbourhood of the  $v$ -th node on  $X_l$ . We will denote this local deformation by  $\mathfrak{R}_{v,l}$ .

Using induction, we fix  $N \geq 0$  and suppose that there is a unique deformation of the multiple root  $\mathfrak{R}$  on  $X_N$  such that for all  $v \leq n'$  this deformation agrees with  $\mathfrak{R}_{v,N}$  around the node  $x_v$ .

Constructing a lift of this deformation to  $X_{n+1}$  and showing that this lift is unique up to unique isomorphism will end the proof. We will do this by fpqc-descent on  $X_{n+1}$ . The synchronized roots around the formal neighbourhoods of the nodes are one portion of the descent data. For the rest of the descent data, we will construct the root away from the nodes and then show compatibility.

On the complement  $W$  of the nodes  $x_1, \dots, x_{n'}$ , the roots we have are all free. Use Lemma 3.9 and Remark 3.10 to conclude that each root deforms uniquely in  $W$ . This uniqueness also proves compatibility with the formal neighbourhoods around the nodes.  $\square$

In Set-up 4.1 we denoted by  $G_v$  the deformation functor of the node  $x_v \in X$ , i.e. of the algebra  $\hat{\mathcal{O}}_{X, x_v} \leftarrow k$ .

Since we need to work with deformations over  $\mathcal{M}$ , even the nodes will not freely deform. The possible deformations of the node  $x_i$  is captured by the image of the map  $D_{\mathcal{M}, p} \rightarrow G_i$ .

Let  $\mathcal{Y} \rightarrow \text{Spec } \hat{\Lambda}$  be the universal deformation of  $X$  over  $\mathcal{M}$ . In other words,  $\tilde{\Lambda} = \hat{\mathcal{O}}_{\mathcal{M}, p}$  and  $\mathcal{Y}$  is the pullback of  $\mathcal{C} \rightarrow \mathcal{M}$ . Let  $U_i = \text{Spec } \hat{\mathcal{O}}_{\mathcal{Y}, x_i}$  be the formal neighbourhood of  $x_i \in \mathcal{Y}$ . Choose a trivialization of  $\mathcal{N}$  on  $U_i$ .

If  $(\mathcal{X}/R, \iota)$  is any deformation of  $X/k$  over  $\mathcal{M}$ , then the formal neighbourhood  $\text{Spec } \hat{\mathcal{O}}_{\mathcal{X}, x_i}$  will factor through  $U_i$ , hence the trivialization of  $\mathcal{N}$  on  $U_i$

pulls back to  $\text{Spec } \hat{\mathcal{O}}_{\mathcal{X}, x_i}$ . In this way, we can identify deformations of roots of the trivial bundle with the deformations of roots of  $\mathcal{N}$  (around the nodes).

Let  $F_v$  denote the deformation functor of the node  $x_v$  and the (generalized) multiple root  $\hat{x}_v^*(\bar{\mathcal{R}}, \bar{\Phi})$ , now viewed as a multiple root of the trivial bundle via our identification. This last point is crucial in actually defining  $F_v$ , because the line bundle  $\mathcal{N}$  can not be made sense of for a general deformation of the node.

Finally, we are ready to state the main technical result we have been building up to since the last three sections.

**Theorem 4.22.** *The map  $D_\xi \rightarrow D_{\mathcal{M}, p} \times_{G_1} F_1 \times_{G_2} F_2 \cdots \times_{G_{n'}} F_{n'}$  is an isomorphism.*

*Proof.* We described the map above, but let us summarize it here. The map  $D_\xi \rightarrow D_{\mathcal{M}, p}$  just forgets the roots. Now, passing to the formal neighbourhood of the  $i$ -th node gives  $D_{\mathcal{M}, p} \rightarrow G_i$ .

Pulling back the tuple of roots to the formal neighbourhood of  $x_i$  gives us a map  $D_\xi \rightarrow F_i$ . To define this map properly, we used trivializations of  $\mathcal{N}$  around the universal deformation of  $X$  over  $\mathcal{M}$  hence the maps  $D_\xi \rightarrow F_i$  are not canonical. Forgetting the root here gives us a map to  $G_i$ .

The main technical difficulty in establishing this result is the construction of the inverse map. This inverse is given in Lemma 4.21.  $\square$

Since all of these functors are pro-representable we conclude that  $D_\xi$  is also pro-representable. In fact, representing  $G_i$  with  $\Lambda[[t_i]]$  and  $F_i$  with  $\Lambda[\tau_i]/(\tau_i^2 - t_i)$  we calculate that  $D_\xi$  is represented by the ring  $T'$  below:

$$T' := \left( \left( \cdots \left( \tilde{\Lambda} \otimes_{\Lambda[[t_1]]} \Lambda[\tau_1]/(\tau_1^2 - t_1) \right) \otimes_{\Lambda[[t_2]]} \cdots \right) \otimes_{\Lambda[[t_{n'}]]} \Lambda[\tau_{n'}]/(\tau_{n'}^2 - t_{n'}) \right).$$

Denoting the image of  $t_i$  in  $\tilde{\Lambda}$  by  $\tilde{t}_i$  we can write this as follows:

$$T' = \tilde{\Lambda}[\tau_1, \dots, \tau_{n'}]/(\tau_1^2 - \tilde{t}_1, \dots, \tau_{n'}^2 - \tilde{t}_{n'}).$$

Recall that  $\tilde{t}_i$  describes how the node  $x_i$  deforms in  $\mathcal{Y} \rightarrow \text{Spec } \tilde{\Lambda}$  in the sense that  $\hat{\mathcal{O}}_{\mathcal{Y}, x_i} \simeq \tilde{\Lambda}[[x, y]]/(xy - \tilde{t}_i)$ .

Furthermore, Lemma 4.21 implies that  $D_\xi$  is not just pro-represented by  $T'$  but this representation is *effective*: there is a universal deformation of  $\xi$  over  $T'$ .

Recalling Set-up 4.19 regarding our convention for  $n'$  we summarize all this in the following statement.

**Corollary 4.23.** *Let  $p: \text{Spec } k \rightarrow \mathcal{M}$  give a pair  $(X, \bar{\mathcal{N}})$  and let  $\xi = (X, \bar{\mathcal{N}}, \bar{\mathcal{R}}, \bar{\Phi})$ . The local deformation functor  $D_\xi$  of  $\xi$  over  $\mathcal{M}$  is pro-represented by*

$$T' = \hat{\mathcal{O}}_{\mathcal{M}, p}[\tau_1, \dots, \tau_{n'}]/(\tau_1^2 - \tilde{t}_1, \dots, \tau_{n'}^2 - \tilde{t}_{n'})$$

*such that the forgetful map  $D_\xi \rightarrow D_{\mathcal{M}, p}$  corresponds to the inclusion  $\hat{\mathcal{O}}_{\mathcal{M}, p} \hookrightarrow T'$ . Moreover, there is a universal deformation of  $\xi$  over  $T'$  making this representation effective.*

Perhaps one of the most important application of this result is to  $\mathcal{M} = \overline{\mathcal{M}}_g$  with the universal curve over it. Then,  $\mathcal{N}$  is  $\omega_{\mathcal{C}/\mathcal{M}}^{\otimes l}$  for some  $l \in \mathbb{Z}$ .

In this case  $\mathcal{O}_{\mathcal{M},p} \simeq T = \Lambda[[t_1, \dots, t_{3g-3}]]$ . Let  $T' = \Lambda[[\tau_1, \dots, \tau_{3g-3}]]$  so that the map  $T \rightarrow T'$  below corresponds to the forgetful functor  $D_\xi \rightarrow D_X$ :

$$T \rightarrow T' : t_i \mapsto \begin{cases} \tau_i^2 & : i \leq n' \\ \tau_i & : i > n' \end{cases} . \quad (4.4.1)$$

These explicit calculations of the representing ring allows us to conclude the following result. We state it in this following weak format since we don't have the appropriate moduli space of multiple roots yet. The complete version is stated in Theorem 5.48.

**Corollary 4.24.** *If the natural map  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  is smooth, then  $D_\xi \rightarrow \text{hom}(\hat{\mathcal{O}}_{S,s}, \_)$  is smooth.*

*Proof.* Recall  $s \in S$  is the image of  $m \in \mathcal{M}$  corresponding to  $X$ . So we need only observe that  $D_\xi$  is represented by the following ring:

$$\hat{\mathcal{O}}_{\mathcal{M},p} \otimes_{\Lambda[[t_1, \dots, t_{3g-3}]]} \Lambda[[\tau_1, \dots, \tau_{3g-3}]]$$

where  $\Lambda[[t_1, \dots, t_{3g-3}]] \rightarrow \hat{\mathcal{O}}_{\mathcal{M},p}$  is the natural map and the other map is defined in Equation 4.4.1 above. Note that we obtain this isomorphism not through some moduli interpretation but rather by calculation.

Recalling that  $\Lambda = \hat{\mathcal{O}}_{S,s}$  the result follows from the observation that smoothness is preserved under pullback and composition.  $\square$

## 5 The moduli space of roots

### 5.1 Single roots

Recall the definition of  $\mathcal{S}(\mathcal{N})$  and  $\overline{\mathcal{S}}(\mathcal{N})$  from Definition 1.12 together with our conventions on  $\mathcal{M} \rightarrow S$  and  $\mathcal{C} \rightarrow \mathcal{M}$  from Section 1.3.

#### 5.1.1 Fundamental properties of the moduli space of roots

Here we list the basic results that can be obtained from Jarvis' work on spin curves [Jar98]. We simply point to the relevant results and make a few remarks that may be particular to our situation.

**Proposition 5.1.**  *$\overline{\mathcal{S}}(\mathcal{N})$  is an algebraic stack.*

*Proof.* We used the hypothesis that  $\mathcal{N}$  has absolutely bounded degree to guarantee the result of Sublemma 4.1.10 [Jar98]. The rest of the proof of Proposition 4.1.7 [Jar98] works without requiring that  $\mathcal{N}$  be the dualizing sheaf and providing us with a smooth cover.

The proof of Proposition 4.1.14 makes no use of the fact that  $\mathcal{N}$  is the dualizing sheaf and applies equally well to our case, proving that the diagonal  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is representable. Since  $\mathcal{M} \rightarrow S$  has representable diagonal, we are done.  $\square$

**Proposition 5.2.** *The diagonal of  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is finite.*

*Proof.* The diagonal is representable and hence of finite type. The diagonal is also unramified by Proposition 4.1.15, whose arguments applies without change to our situation.

To finish the proof we need to show that the diagonal is proper. This is done using the valuative criterion and working with a complete DVR as the base.

In this case the automorphisms come in two flavors as described in Lemma 2.22. Either the automorphism is multiplication by  $\pm 1$  or for each node that persists over the entire base we have an additional automorphism of order 2.

There is no obstacle to extending these additional automorphisms as they are also defined by  $\pm 1$ . Which means that any automorphism on the generic fiber extends to the special fiber.  $\square$

**Proposition 5.3.**  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is proper and of finite type.

*Proof.* The proof of Lemma 4.1.8 [Jar98] constructs a smooth atlas of  $\overline{\mathcal{S}}(\mathcal{N})$  and it is easy to see that this atlas is of finite type over the curve  $\mathcal{C} \rightarrow \mathcal{M}$ .

The properness of this map follows from §4.2.2 *loc.cit.* when the fibers of  $\mathcal{C} \rightarrow \mathcal{M}$  are generically smooth. When  $\mathcal{C} \rightarrow \mathcal{M}$  consists entirely of singular curves we can do the following.

Let  $R$  be a DVR with quotient field  $K$ . Suppose we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec} K & \longrightarrow & \overline{\mathcal{S}}(\mathcal{N}) \\ \downarrow & & \downarrow \\ \mathrm{Spec} R & \longrightarrow & \mathcal{M} \end{array}$$

Let  $(C_R \rightarrow \mathrm{Spec} R, \mathcal{N}_R)$  be obtained from the bottom horizontal map and  $(\mathcal{E}_K, b_K: \mathcal{E}_K^2 \rightarrow \mathcal{N}_K)$  be the root obtained from the top horizontal map. Normalize the curve  $C_R \rightarrow \mathrm{Spec} R$ . For notational convenience we will assume there are only two component in this normalization, say  $Y_1, Y_2$ , with  $Z_i \subset Y_i$  glued together to form  $C_R$ . Let  $z_i = Y_i|_K \cap Z_i$ .

Pulling back  $(\mathcal{E}_K, b_K)$  to  $Y_i|_K$  gives a root of  $\mathcal{N}|_{Y_i|_K}(-z_i)$ , see §4.1.4.1 [Jar98] for more about roots on curves over a field. Then, applying §4.2.2 *loc.cit.* to each of these components but using the twisted line bundles  $\mathcal{N}|_{Y_i}(-Z_i)$ , we can extend our root to these components (possibly after a finite base change). Note that this extension is unique.

Let  $(\mathcal{E}_i, b_i)$  be the root of  $\mathcal{N}|_{Y_i}(Z_i)$  on  $Y_i$  obtained by extending  $(\mathcal{E}_K, b_K)|_{Y_i|_K}$ . Let  $\pi_i: Y_i \rightarrow C_R$  be the inclusion maps. The torsion-free sheaf underlying our root will be  $\mathcal{E} := \pi_{1,*}\mathcal{E}_1 \oplus \pi_{2,*}\mathcal{E}_2$ .

Each  $b_i$  induces a map  $b'_i := \pi_{i,*}b_i: \pi_{i,*}\mathcal{E}_i^2 \rightarrow \pi_{i,*}\mathcal{N}|_{Y_i}(-Z_i) \hookrightarrow \mathcal{N}$  so that we may let  $b: \mathcal{E}^2 \rightarrow \mathcal{N}$  be the map which agrees with  $b'_i$  for  $i = 1, 2$ . Note that,  $b$  is the zero map along  $Z$ .

We need only show that this root is unique to finish the proof. This follows from the classification of roots near singularities. Since  $R$  is integral, the singularity is of the form  $R[[x, y]]/(xy)$ . Then the only roots around this singularity are of the form  $(E(0, 0), s)$ . This agrees with our construction.  $\square$

**Remark 5.4.** Jarvis claims that the moduli space of  $r$ -th roots is *finite* over the moduli space of curves. This is not true since each root has a non-trivial automorphism over the curve (namely multiplication by constants whose  $r$ -th power is 1). Therefore this morphism is not representable, hence not finite.

**Definition 5.5.** If a morphism  $f: \mathcal{X} \rightarrow \mathcal{Y}$  of algebraic stacks has unramified diagonal  $\Delta_f: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$  then let us call  $\mathcal{X}$  a *relatively DM stack over  $\mathcal{Y}$* . The morphism  $f$  is then called a DM stack.

**Remark 5.6.** In the footnote 1 of [Stacks, Tag 04YV] it is shown that the fibers over schemes of a relatively DM stack are DM stacks.

**Corollary 5.7.**  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow S$  is a Deligne–Mumford stack if  $\mathcal{M}$  is a Deligne–Mumford stack.

*Proof.* Since  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  has unramified diagonal,  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is relatively DM by definition. Being relatively DM is stable under composition and being relatively DM over a scheme is equivalent to being DM. See [Stacks, Tag 04YV] for proofs of these statements.  $\square$

**Proposition 5.8.** If the natural map  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  is smooth then  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow S$  is smooth.

*Proof.* We have not defined multiple roots for families but when  $m = 1$  we can still use Corollary 4.24 from which the result is immediate.  $\square$

**Proposition 5.9.**  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is quasi-finite.

*Proof.* See §4.1.4.1 of [Jar98] where the isomorphism classes of roots on a curve over a field are identified with isomorphism classes of line bundles squaring to a fixed line bundle on partial normalizations. The latter set is finite hence we are done.  $\square$

### 5.1.2 Basic properties regarding families of roots

Given a family of curves  $C \rightarrow T$ , the individual singularities of each fiber are referred to as the node of a fiber. But the locus of these nodes are a well defined closed subscheme of  $C$  and we use the following terminology to distinguish this locus from individual nodes.

**Definition 5.10.** For a family of nodal curves  $C \rightarrow T$ , the closed subscheme  $Z \subset C$  cutout by the first fitting ideal of the relative dualizing sheaf  $\Omega_{C/T}$  is called the *discriminant locus of the curve  $C/T$* .

**Remark 5.11.** If  $T' \rightarrow T$  and  $C' = C \times_T T' \rightarrow T'$  is the pullback curve then the discriminant locus  $Z'$  of the curve  $C'/T'$  coincides with the pullback  $Z \times_C C'$  of the discriminant locus  $Z$  of  $C/T$ . For this reason we say the discriminant locus behaves well with respect to base change.

Recall that the torsion-free module underlying a root is locally self-dual. The following observation is useful in visualizing where the difficulties lie in trying to synchronize a pair of roots: it is never an arbitrary part of the discriminant locus but a union of connected components of it.

**Lemma 5.12.** Let  $C \rightarrow T$  be a stable curve and  $\mathcal{E}$  a locally self-dual rank-1 torsion-free module on the curve. Then the rank of  $\mathcal{E}$  is constant on each component of the discriminant locus.

*Proof.* Pick any point  $\mathfrak{p} \in Z$ . The module  $\mathcal{E}$  has either rank 1 or rank 2 at  $\mathfrak{p}$ . We will show both the properties of being rank 1 and rank 2 are open in  $Z$  around  $\mathfrak{p}$ . By semi-continuity, being of rank 1 is an open condition so it remains to show the latter.

First perform an étale base change if necessary, and then pass to an étale neighbourhood  $U \rightarrow C$  of  $\mathfrak{p}$  so that may assume  $B = \text{Spec } R$  and  $U = \text{Spec } A$  where  $\exists x, y \in A$ ,  $\pi \in R$  such that  $Z_U$  is the vanishing set of  $(x, y)$  and  $\pi = xy$ . See Remark 5.39.

By choosing  $U$  appropriately, we may apply Faltings' classification [Fal96] and conclude that  $E := \mathcal{E}_U$  is either free or of the form  $E(p, p)$  where  $p^2 = \pi$ . Note that being free is an open condition, so that we need only consider the latter condition.

Note  $Z_U$  is isomorphic to  $\text{Spec } R/(\pi)$  and  $p^2 = \pi$ . Now  $E(p, p)$  is free at a point of  $Z_U$  iff  $p$  is invertible at that point. However, this is impossible in  $R/(\pi)$  since  $p^2 = 0 \pmod{(\pi)}$ . This proves that being of rank 2 (along the node) is open around  $\mathfrak{p}$ .  $\square$

## 5.2 Multiple roots

Let  $T$  be a scheme,  $C \rightarrow T$  a stable curve and  $\mathcal{N}$  a line bundle on  $C$ . Let  $\mathcal{R} = (\mathcal{E}_i, b_i)_{i=1}^m$  a sequence of roots of  $\mathcal{N}$ .

**Definition 5.13.** For each  $T' \rightarrow T$ , and for each  $i = 1, \dots, m$ , define  $V_i(T') \subset C' := C \times_T T'$  to be the locus of points  $x$  where the rank of  $\mathcal{E}_i$  is maximal among all  $\mathcal{E}_j$ . In symbols,

$$\dim_{\kappa(x)} \mathcal{E}_i|_x = \max\{\dim_{\kappa(x)} \mathcal{E}_j|_x \mid j = 1, \dots, m\}.$$

**Remark 5.14.** In light of Lemma 5.12 the locus  $V_i(T')$  is open: it is the complement of a finite number of components of the discriminant locus, where  $\mathcal{E}_i$  is free but some  $\mathcal{E}_j$  are not.

We will first state the definition of a multiple root which has the advantage of capturing the geometric interpretation most readily. See Remark 5.16 for a summary of this interpretation. In Section 5.2.1 we will state and prove alternative definitions relating it to our previous work.

**Definition 5.15** (Multiple Root). Let  $D$  be a sheaf of graded algebras on  $C$  and let  $\Psi = (\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i)_{i=1}^m$  be a sequence of graded morphisms such that

- for all  $i, j$  we have  $\text{Sym } b_i \circ \psi_i = \text{Sym } b_j \circ \psi_j$ ,
- for all  $i$  the map  $\psi_i|_{V_i}$  is an isomorphism.

Then the tuple  $(\mathcal{R}, \Psi)$  will be called a *multiple root*.

**Remark 5.16.** The motivation for this definition can be summarized as follows. Each root  $\mathcal{E}_i$  gives rise to a space  $\mathbb{P}(\mathcal{E}_i) \rightarrow C$  together with a line bundle  $\mathcal{L}_i$  on  $\mathbb{P}(\mathcal{E}_i)$  which pushes forward to  $\mathcal{E}_i$  on  $C$ . The definition above identifies these  $\mathbb{P}(\mathcal{E}_i)$  wherever possible as well as the line bundles  $\mathcal{L}_i^{\otimes 2}$ , when this makes sense. For a more comprehensive treatment, see Appendix A.

**Remark 5.17.** Note that  $D$  is locally generated in degree 1, because  $\mathrm{Sym}^{2*} \mathcal{E}_i$  are locally generated in degree 1.

Let  $\mathcal{M} \rightarrow S$  be a DM stack, locally of finite type over an excellent scheme  $S \rightarrow \mathrm{Spec} \mathbb{Z}[1/2]$ . Let  $\mathcal{C} \rightarrow \mathcal{M}$  be a stable curve over  $\mathcal{M}$  and  $\mathcal{N}$  a line bundle on  $\mathcal{C}$ .

**Definition 5.18.** Let  $\overline{\mathcal{S}}^m(\mathcal{N})$  be the category fibered in groupoids defined over  $\mathcal{M}$  whose fiber over  $T \rightarrow \mathcal{M}$  is the groupoid of multiple roots of  $\mathcal{N}_B$  on  $\mathcal{C}_B := \mathcal{C} \times_{\mathcal{M}} B$ .

Now we give a definition that is more convenient to study general properties of  $\overline{\mathcal{S}}^m(\mathcal{N})$ .

### 5.2.1 Working definition for multiple roots

The symmetric algebra generated by the roots is not a finitely presented module, which makes it inconvenient to work with. We will now rectify this problem.

**Definition 5.19.** Let  $\mathcal{F}$  be a sheaf of modules on  $C$  and let  $\Phi := (\varphi_i: \mathcal{F} \rightarrow \mathcal{E}_i^2)_{i=1}^m$  be a sequence of maps such that

- for all  $i, j$  we have  $b_i \circ \varphi_i = b_j \circ \varphi_j$ ,
- for all  $i$  the map  $\varphi_i|_{V_i}$  is an isomorphism.

Then we will call  $\Phi$  a *pre-sync data* and the pair  $(\mathcal{R}, \Phi)$  will be called a *pre-synced tuple of roots*. We will refer to these two conditions as *pre-sync conditions*.

**Remark 5.20.** The maps  $b_i: \mathcal{E}_i^2 \rightarrow \mathcal{N}$  glue to a map  $b_\Phi: \mathcal{F} \rightarrow \mathcal{N}$ .

On  $V_{ij} := V_i \cap V_j$  we can define  $\psi_{ij} = \varphi_j|_{V_{ij}} \circ \varphi_i|_{V_{ij}}^{-1}: \mathcal{E}_i^2 \xrightarrow{\sim} \mathcal{E}_j^2$ . Using  $\psi_{ij}$ , we get a surjective map

$$\mathrm{Sym}^* \mathrm{Sym}^2 \mathcal{E}_i|_{V_{ij}} \rightarrow \mathrm{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}.$$

We would like this to factor through an isomorphism  $\mathrm{Sym}^{2*} \mathcal{E}_i|_{V_{ij}} \xrightarrow{\sim} \mathrm{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}$ , we give it a name when this happens.

**Definition 5.21.** If for each tuple  $i, j$  the isomorphism  $\psi_{ij}$  induces an isomorphism  $\mathrm{Sym}^{2*} \mathcal{E}_i|_{V_{ij}} \xrightarrow{\sim} \mathrm{Sym}^{2*} \mathcal{E}_j|_{V_{ij}}$  then we say  $\Phi$  is a *sync data*. The tuple  $(\mathcal{R}, \Phi)$  is then called a *synced tuple of roots*. This condition will be called the *sync condition*.

**Lemma 5.22.** *The sync condition holds whenever the morphism*

$$\mathrm{Sym}^2 \mathrm{Sym}^2 \mathcal{E}_i|_{V_{ij}} \rightarrow \mathrm{Sym}^4 \mathcal{E}_j|_{V_{ij}}$$

*factors through  $\mathrm{Sym}^4 \mathcal{E}_i|_{V_{ij}} \rightarrow \mathrm{Sym}^4 \mathcal{E}_j|_{V_{ij}}$ .*

*Proof.* This follows because the kernel of  $\mathrm{Sym}^* \mathrm{Sym}^2 \mathcal{E}_i \rightarrow \mathrm{Sym}^{2*} \mathcal{E}_i$  is generated by the kernel of  $\mathrm{Sym}^2 \mathrm{Sym}^2 \mathcal{E}_i \rightarrow \mathrm{Sym}^4 \mathcal{E}_i$ .  $\square$

For a pair of pre-synced or synced roots the definition of an isomorphism is the natural one: a sequence of isomorphisms between the roots which lift to  $D$ 's or  $\mathcal{F}$ 's respectively. We define only one here.



**Definition 5.23.** An isomorphism between a pair of synced roots  $\mathfrak{R} = (\mathcal{R}, \Phi)$  and  $\mathfrak{R}' = (\mathcal{R}', \Phi')$  is a sequence of isomorphisms  $\mu = (\mu_i)_{i=1}^m: \mathcal{R} \rightarrow \mathcal{R}'$  of the underlying roots, admitting an isomorphism  $\mu_0: \mathcal{F} \xrightarrow{\sim} \mathcal{F}'$  commuting with the synchronizations.

**Remark 5.24.** Notice that, whenever the morphism  $\mu_0$  exists, it is uniquely defined by  $\mu = (\mu_i)_{i=1}^m$ .

**Lemma 5.25.** *The category of synced tuples of roots on  $C$  and the category of multiple roots are equivalent.*

*Proof.* Starting from  $(\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i)$  we define  $\mathcal{F} := D_1$ ,  $\varphi_i := \psi_{i,1}: \mathcal{F} \rightarrow \mathcal{E}_i^2$ . This is clearly a pre-sync data and to see that this is a sync data we need only recall that  $D$  is locally generated in degree 1 and that  $D_2|_{V_i} \xrightarrow{\sim} \text{Sym}^4 \mathcal{E}_i|_{V_i}$ .

For the converse, we first need to define  $D$ . This is done by gluing together  $\text{Sym}^{2*} \mathcal{E}_i|_{V_i}$  on each  $V_i$ . To perform this gluing we note that  $\psi_{ij}$  induces an isomorphism between the symmetric algebras by the definition of sync data. The cocycle condition is satisfied because  $\psi_{ij}$  is locally of the form  $\varphi_j \circ \varphi_i^{-1}$ .

To define the maps  $\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i$ , we begin with the isomorphisms  $D|_{V_i} \xrightarrow{\sim} \text{Sym}^{2*} \mathcal{E}_i|_{V_i}$ . For any  $p$  where  $\mathcal{E}_i$  is free, e.g. for any  $p \notin V_i$ , find  $j$  such that  $p \in V_j$  and choose an open neighbourhood  $U$  of  $p$  such that  $U \subset V_j$  and  $\mathcal{E}_i$  is free on  $U$ . Then we observe that  $b_i$ , and  $\text{Sym} b_i$ , are isomorphisms on  $U$ . So we may now define

$$D|_U \rightarrow \text{Sym}^{2*} \mathcal{E}_j|_U \xrightarrow{b_j} \text{Sym}^* \mathcal{N} \xrightarrow{b_i^{-1}} \text{Sym}^{2*} \mathcal{E}_i|_U.$$

Because  $b_j \circ \varphi_j = b_k \circ \varphi_k$ , the map  $D|_U \rightarrow \text{Sym}^{2*} \mathcal{E}_i|_U$  is independent of our choice of  $j$ . In particular, these maps glue together to give  $\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i$ .  $\square$

### 5.3 Algebraicity

Our goal in this section is to show that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is an algebraic stack and establish its basic properties. We know that  $\overline{\mathcal{S}}(\mathcal{N})$  is an algebraic stack (see Proposition 5.1) and we will prove that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is an algebraic stack by induction on  $m \geq 1$ . We now prepare to make this induction step possible.

**Lemma 5.26.** *There is a canonical forgetful functor  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}^{m-1}(\mathcal{N})$  forgetting the  $m$ -th root and adjusting the sync data appropriately.*

*Proof.* Let  $C \rightarrow T$  be a stable curve, with  $\mathcal{R} = (\mathcal{E}_i, b_i)_{i=1}^m$  and  $\Phi = (\varphi_i: \mathcal{F} \rightarrow \mathcal{E}_i^2)_{i=1}^m$  giving a synchronized tuple of roots. We want to define a sync data  $\Phi' = (\varphi'_i: \mathcal{F}' \rightarrow \mathcal{E}_i^2)_{i=1}^{m-1}$  for the tuple  $\mathcal{R}' := (\mathcal{E}_i, b_i)_{i=1}^{m-1}$ .

Recall that for each  $i = 1, \dots, m$  we defined  $V_i$  to be the locus in  $C$  for which  $\mathcal{E}_i$  has maximal rank among the  $m$  roots. Similarly, for each  $i = 1, \dots, m-1$  define  $W_i$  to be the locus in  $C$  for which  $\mathcal{E}_i$  has maximal rank among the first  $m-1$  roots. Clearly  $V_i \subset W_i$ .

We will define  $\mathcal{F}'$  by gluing  $\mathcal{E}_i^2|_{W_i}$  together for  $i = 1, \dots, m-1$ . Let  $Z_{ij} \subset C$  be the closed locus in which  $\mathcal{E}_i$  and  $\mathcal{E}_j$  are both non-free. Clearly  $Z_{ij} \subset V_i \cap V_j \subset W_i \cap W_j$ . Outside of  $Z_{ij}$  the maps  $b_i$  and  $b_j$  are isomorphisms, so the isomorphism  $\varphi_{ij}: \mathcal{E}_i^2|_{V_{ij}} \xrightarrow{\sim} \mathcal{E}_j^2|_{V_{ij}}$  naturally extends to all of  $W_{ij} = W_i \cap W_j$  as  $b_j^{-1} \circ b_i$ . This gluing procedure defines  $\varphi_i: \mathcal{F}' \rightarrow \mathcal{E}_i^2$  for all  $i = 1, \dots, m-1$  and it is easy to check that this is a sync data.  $\square$

**Remark 5.27.** For any proper subset  $J \subset \{1, \dots, m\}$  we can forget the roots indexed by  $J$  and modify the synchronizations appropriately, however we will not use this except for the simple case where  $J = \{1, \dots, m-1\}$ . This defines a map  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}(\mathcal{N})$  forgetting all but the  $m$ -th root. Now combine the two maps, the first map forgetting the  $m$ -th root and the second map forgetting all roots except the  $m$ -th root. We obtain a map  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ . In effect, this map forgets how the  $m$ -th root is synchronized with the rest of the roots.

Take  $B \rightarrow \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  and denote the corresponding data by  $(C \rightarrow T, \mathcal{N}, \mathfrak{R}', \mathcal{R}_m)$  where  $\mathfrak{R}' = ((\mathcal{R}_i)_{i=1}^{m-1}, \Phi' = (\varphi'_i: \mathcal{F} \rightarrow \mathcal{E}_i))$  is an  $m-1$  tuple of synced roots of  $\mathcal{N}$ , and  $\mathcal{R}_m$  a root of  $\mathcal{N}$ . As usual  $\mathcal{R}_i = (\mathcal{E}_i, b_i)$ .

Let  $\mathcal{F}$  be a coherent sheaf of modules on  $C$  and let  $\tau_1: \mathcal{F} \rightarrow \mathcal{F}'$ ,  $\tau_2: \mathcal{F} \rightarrow \mathcal{E}_m^2$  be morphisms. Define  $\varphi_i: \mathcal{F} \rightarrow \mathcal{E}_i^2$  by  $\varphi_i = \varphi'_i \circ \tau_1$  if  $i < m$  and by  $\varphi_m = \tau_2$  otherwise. Denote the  $m$ -tuple of roots by  $\mathcal{R} = (\mathcal{R}_i)_{i=1}^m$ .

**Definition 5.28.** If  $\Phi$  is a (pre-)sync data for  $\mathcal{R}$  then we will call  $(\tau_1, \tau_2)$  a (pre-)sync data for  $\mathfrak{R}'$  and  $\mathcal{R}_m$ .

**Theorem 5.29.**  $\overline{\mathcal{S}}^m(\mathcal{N})$  is an algebraic stack, locally of finite type over  $S$ .

*Proof.* We will use induction on  $m$ . The base case  $m = 1$  is Proposition 5.1 and Proposition 5.3 so we assume  $m \geq 2$ .

Let  $\mathcal{Y} = \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  and consider the forgetful map  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{Y}$  described in Remark 5.27.

By induction hypothesis,  $\mathcal{Y}$  is an algebraic stack, locally of finite type over  $S$ , and so we need to show that for any  $\mathcal{Y}$ -scheme  $B$  the pullback  $\overline{\mathcal{S}}^m(\mathcal{N})_B \rightarrow B$  is an algebraic stack locally of finite over  $B$ . We will do this by carving out  $\overline{\mathcal{S}}^m(\mathcal{N})_B$  from an ambient stack.

**The ambient stack:** Take  $B \rightarrow \mathcal{Y}$ . This defines a curve  $C \rightarrow B$  and roots  $(\mathcal{E}_i, b_i)_{i=1}^{m-1}$  together with a sync data  $\Phi' = (\varphi'_i: \mathcal{F}' \rightarrow \mathcal{E}_i^2)_{i=1}^{m-1}$ . In addition we have a root  $(\mathcal{E}_m, b_m)$  on  $C$ . For any  $B' \rightarrow B$  denote by  $\mathcal{F}'_{B'}$  the pullback of  $\mathcal{F}'$  to  $C_{B'} := C \times_B B'$ , similarly define  $\varphi_i|_{B'}$  and  $\mathcal{E}_i|_{B'}$ . Denote by  $b': \mathcal{F}' \rightarrow \mathcal{N}_B$  the pullback of the map  $b_{\Phi'}: \mathcal{F} \rightarrow \mathcal{N}$ .

Let  $\mathcal{A}' \rightarrow B$  be the category of tuples  $(B' \rightarrow B, \mathcal{F})$  where  $\mathcal{F}$  is a quasi-coherent sheaf on  $C_{B'}$  which is  $B'$ -flat, finitely presented and has  $B'$ -proper support. In [Hal] it is shown that  $\mathcal{A}'$  is an algebraic stack, locally of finite type over  $B$ .

Let  $\mathcal{A} \rightarrow \mathcal{A}'$  be the category of tuples  $(B' \rightarrow B, \mathcal{F}, (\tau_1: \mathcal{F} \rightarrow \mathcal{F}'_{B'}, \tau_2: \mathcal{F} \rightarrow \mathcal{E}_m|_{B'}))$  with  $(B' \rightarrow B, \mathcal{F}) \in \mathcal{A}'$  and  $b'|_{B'} \circ \tau_1 = b_m|_{B'} \circ \tau_2$ . Since all the relevant modules are finitely presented,  $\mathcal{A}$  is an algebraic stack and will serve as our ambient stack.

It is clear that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is a subcategory of  $\mathcal{A}$  and we will now show that the inclusion  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{A}$  is a locally closed immersion.

Let  $\mathcal{A}_1 \subset \mathcal{A}$  be the subcategory of  $\mathcal{A}$  for which  $\tau_i$ 's satisfy the pre-sync condition. Let  $\mathcal{A}_2 \subset \mathcal{A}_1$  be the subcategory of  $\mathcal{A}_1$  for which the  $\tau_i$ 's satisfy the sync condition. Note  $\mathcal{A}_2 = \overline{\mathcal{S}}^m(\mathcal{N})$ .

We are done once we prove that  $\mathcal{A}_1 \rightarrow \mathcal{A}$  is an open immersion and  $\mathcal{A}_2 \rightarrow \mathcal{A}_1$  a closed immersion.

**Pre-sync condition is open:** The proof that  $\mathcal{A}_1 \rightarrow \mathcal{A}$  is an open immersion takes all of Section 5.3.1 culminating in Proposition 5.34.

**Sync condition is closed:** The proof that  $\mathcal{A}_2 \rightarrow \mathcal{A}_1$  is a closed immersion takes place in Section 5.3.2, see Proposition 5.35.  $\square$

### 5.3.1 Pre-sync condition is open

**Definition 5.30.** For any  $T \rightarrow \mathcal{A}$  we will define  $U_i(T), V_i(T) \subset C_T$  for  $i = 1, 2$  as follows. Let  $U_i(T) = \{x \in C_T \mid \tau_i|_x \text{ is an isomorphism}\}$ . Let  $V_1(T)$  be the complement of the loci where  $\mathcal{F}'$  is free but  $\mathcal{E}_m$  is not. Let  $V_2(T)$  be the complement of the loci where  $\mathcal{E}_m$  is free but  $\mathcal{F}'$  is not.

**Remark 5.31.** Whenever we are working locally on  $C$ , we may assume  $m = 2$  since  $\mathcal{F}'$  is locally isomorphic to one of  $\mathcal{E}_i^2$ . When  $m = 2$  then  $\mathcal{F}' = \mathcal{E}_1^2$ .

With this remark in mind we will assume that there is a pair of maps  $(\varphi_i: \mathcal{F} \rightarrow \mathcal{E}_i^2)_{i=1}^2$  satisfying  $b_1 \circ \varphi_1 = b_2 \circ \varphi_2$  and that our goal is to show the second pre-sync condition defines an open locus on the base  $T$ . Note that the second pre-sync condition is equivalent to having  $U_i = V_i$ .

**Lemma 5.32.** *The sets  $U_i$  and  $V_i$  are open. Furthermore, they respect base change. More precisely, for any  $S \rightarrow T \rightarrow \mathcal{A}$  we have:*

- $V_i(S) = V_i(T)|_S$
- $U_i(S) = U_i(T)|_S$

*Proof.* The complement of  $V_i$  is the locus of points for which  $\mathcal{E}_i$  is free but  $\mathcal{E}_j$  is not ( $j \neq i$ ). This locus is supported on the discriminant locus. But we showed in Lemma 5.12 that the rank of a root is constant on each connected component of the discriminant locus. Thus  $V_i^c$  is a union of components of the discriminant locus, which is closed.

Moreover, the condition of being locally free or not behaves well with respect to base change. Therefore it is clear that  $V_i(S) = V_i(T)|_S$ .

The fact that  $U_i$ 's respect base change and are open follows from a general fact. Let  $\psi: F \rightarrow E$  be a map of finitely presented modules on  $C_T$ . Then the set where  $\psi$  is an isomorphism is the intersection  $\{\ker \psi = 0\} \cap \{\text{coker } \psi = 0\}$ . When  $E$  is flat over  $T$  then for any  $T' \rightarrow T$  we have  $\{(\ker \psi)|_{T'} = 0\} \cap \{(\text{coker } \psi)|_{T'} = 0\} = \{\ker(\psi|_{T'}) = 0\} \cap \{\text{coker}(\psi|_{T'}) = 0\}$ .

It is a standard fact that the zero set of a finitely generated module is open and respects base change. Hence we are done.  $\square$

**Lemma 5.33.** *If  $V_1 \subset U_1$  then  $U_2 \subset V_2$ . Similarly with the indices swapped. Thus  $V_i \subset U_i$  for both  $i = 1, 2$  implies  $V_i = U_i$  for both  $i = 1, 2$ .*

*Proof.* Assuming  $V_1 \subset U_1$  we have  $V_2^c \subset V_1 \subset U_1$  by definitions. Therefore, if  $\exists x \in V_2^c \cap U_2$  then  $x \in U_1 \cap U_2$ . But this is a contradiction, if  $\varphi_1$  and  $\varphi_2$  are isomorphisms at  $x$  then  $\mathcal{E}_i$ 's are isomorphic at  $x$ . On the other hand  $x \in V_2^c$  implies that the roots have different ranks at  $x$ .  $\square$

The following proves that  $\mathcal{A}_1 \rightarrow \mathcal{A}$  is an open immersion.

**Proposition 5.34.** *Take a map  $\text{Spec } R \rightarrow \mathcal{A}$ . Let  $\mathfrak{p} \in \text{Spec } R$  be a point such that  $U_i(k) = V_i(k)$  for  $i = 1, 2$  where  $k$  is the residue field of  $\mathfrak{p}$ . Then there exists a Zariski open neighbourhood  $W$  of  $\mathfrak{p}$  such that  $U_i(W) = V_i(W)$  for  $i = 1, 2$ .*

*Proof.* Let  $j: C|_{\mathfrak{p}} \hookrightarrow C$  be the inclusion of the fiber over  $\mathfrak{p}$ . We will refer to  $C|_{\mathfrak{p}}$  as the central fiber even though  $R$  is not necessarily local.

The fact that  $U_i(k) = U_i(R)|_{\text{Spec } k}$  implies that  $U_i(R)$  is an open neighbourhood of  $j(U_i(k))$ , similarly for  $V_i$ 's. We know  $V_i(k)$ 's cover the central fiber

and, by hypothesis,  $U_i(k) = V_i(k)$ . Therefore the open sets  $U_i(R) \cap V_i(R)$  for  $i = 1, 2$  cover the central fiber in  $C$ .

Pick a Zariski neighbourhood  $W \subset \text{Spec } R$  of  $\mathfrak{p}$  such that the preimage of  $W$  is covered by  $U_i(R) \cap V_i(R)$ . Shrink  $W$  so that every component of the discriminant locus intersect the fiber over  $\mathfrak{p}$ . Let  $Z$  be the components of the discriminant locus on which  $\mathcal{E}_i$ 's are both non-free. On  $Z|_{\mathfrak{p}}$  the  $\varphi_i$ 's are isomorphisms, hence they will remain an isomorphism in a neighbourhood of  $Z \cap C_{\mathfrak{p}} \subset C$ . Shrink  $W$  one last time so that  $\varphi_i$ 's are isomorphisms on all of  $Z$ .

Now we claim that  $U_i(W) = V_i(W)$ . By Lemma 5.33 it will be sufficient to show  $V_i(W) \subset U_i(W)$  for  $i = 1, 2$ .

Pick  $x \in V_1(W)$ . We want to show  $x \in U_1(W)$ . If  $x \in U_1(W) \cap V_1(W)$  then we are done. Otherwise,  $x \in U_2(W) \cap V_2(W)$ . Note  $x \in V_1(W) \cap V_2(W)$  implies that either both  $\mathcal{E}_i$ 's are free or both  $\mathcal{E}_i$  are non-free at  $x$ . Furthermore,  $x \in U_2(W)$  implies  $\varphi_2$  is an isomorphism at  $x$ .

If both  $\mathcal{E}_i$ 's are free then the fact that  $\varphi_i$ 's commute with  $b_i$ 's imply that  $\varphi_1$  is also an isomorphism. Hence  $x \in U_1(W)$ .

If  $\mathcal{E}_i$ 's are both non-free, then  $x \in Z$ . But, by our construction of  $W$ ,  $x \in Z$  implies that  $\varphi_1$  is an isomorphism at  $x$ .  $\square$

### 5.3.2 Sync condition is closed

We now want to prove the following proposition. Its proof lasts until the end of Lemma 5.41.

**Proposition 5.35.** *The map  $\mathcal{A}_2 \rightarrow \mathcal{A}_1$  is a closed immersion.*

Pick a morphism from a scheme  $T \rightarrow \mathcal{A}_1$ . Suppose a point  $t \in T$  is such that  $t \rightarrow \mathcal{A}_1$  factors through  $\mathcal{A}_2 \rightarrow \mathcal{A}_1$ . It will suffice to show  $\mathcal{A}_2 \times_{\mathcal{A}_1} T \rightarrow T$  is a closed immersion in an étale neighbourhood of  $t$ .

We will rely heavily on formal neighbourhoods near the discriminant locus. To be able to transfer information from formal neighbourhoods we need to be able to assume that  $T$  is locally noetherian. This is possible if  $\mathcal{A}_1$  is locally noetherian.

**Lemma 5.36.**  *$\mathcal{A}_1 \rightarrow S$  is locally of finite type. In particular,  $\mathcal{A}_1$  is locally noetherian.*

*Proof.* We have the following chain of maps:

$$\mathcal{A}_1 \rightarrow \mathcal{X} \rightarrow \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M} \rightarrow S.$$

The map  $\mathcal{A}_2 \rightarrow \mathcal{X}$  is an open immersion as proved in Proposition 5.34. The map  $\mathcal{Y} := \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N}) \rightarrow S$  is locally of finite type, as part of our induction hypothesis, and the map  $\mathcal{X} \rightarrow \mathcal{Y}$  is locally of finite type as shown in [Hal]. Thus the composition  $\mathcal{A}_1 \rightarrow S$  is locally of finite type.

Since  $S$  is excellent, it is noetherian. Thus  $\mathcal{A}_1$  is locally noetherian.  $\square$

Using the lemma above and by passing to an étale neighbourhood of  $t$ , we may assume  $T = \text{Spec } R$  is noetherian affine and that the discriminant locus of the curve  $C_T \rightarrow T$  has  $n$  components  $Z_1, \dots, Z_n$  such that  $Z_i \cap C_t = v_i$  where  $v_i \in C_t$  is a single node of  $C_t$ .

As we mentioned in Remark 5.31 we may assume  $m = 2$  whenever our constructions are local in  $C$ . This being the case for the rest of this section, from now on we will assume  $m = 2$ .

Denote by  $(\mathcal{E}_i, b_i)_{i=1}^2$  the two roots on  $C_T$  and let  $(\tau_i: \mathcal{F} \rightarrow \mathcal{E}_i^2)_{i=1}^2$  be the pre-sync data associated to the map  $T \rightarrow \mathcal{A}_1$ .

**Definition 5.37.** Let  $F_i \subset \text{hom}(\_, (T, t))$  be the subfunctor defined so that  $(T', t') \rightarrow (T, t)$  belongs to  $F_i$  iff  $(\tau_1, \tau_2)|_{T'}$  gives a sync data around  $Z_i|_{T'}$ .

We only need to check the sync condition in a neighbourhood of each  $Z_i$ . For this reason, the morphism  $\mathcal{A}_2 \times_{\mathcal{A}_1} T \rightarrow T$  can be viewed as a fiber product  $F_1 \times_T \cdots \times_T F_n \rightarrow T$ .

We have thus reduced our goal to proving that  $F_i \rightarrow T$  is a closed immersion for all  $i$ . If one of the two roots is free around  $Z_i$  then this problem is trivial since  $F_i = \text{hom}(\_, T)$ .

So, fix a node  $v_i \in C_t$  where both roots are non-free. At this point, we may assume that there is a single node  $v = v_i \in C_T$  and the discriminant locus  $Z = Z_i$  is connected. Let  $F$  stand for  $F_i$ .

In a Zariski neighbourhood  $V$  of  $Z$  both the morphisms  $\tau_1$  and  $\tau_2$  are isomorphisms, hence we can define  $\psi := \tau_2 \circ \tau_1^{-1}$  on  $V$ . Our goal can then be roughly described as finding the conditions on the base  $T$  for which  $\psi$  induces an isomorphism  $\text{Sym}^{2*} \mathcal{E}_1|_V \xrightarrow{\sim} \text{Sym}^{2*} \mathcal{E}_2|_V$ .

By passing to an étale neighbourhood of  $t$  in  $T$  if necessary, we may assume the existence of an étale neighbourhood  $U \rightarrow C_T$  of  $v$  satisfying the following three conditions:

- $U$  is affine, i.e.,  $U = \text{Spec } A$ .
- The image of  $U \rightarrow C_T$  contains  $Z$ .
- For each  $i = 1, 2$  we have  $(\mathcal{E}_i, b_i)|_U \simeq (E(p_i, p_i), s)$  for some  $p_i \in R$ .

Here  $E(p_i, p_i)$  is defined as in Section 2.4.1. For the possibility of assuming the third condition, see Theorem 3.9 [Fal96].

**Definition 5.38.** An étale neighbourhood  $U \rightarrow C_T$  of  $Z$  satisfying the three conditions above will be called a *preferred neighbourhood* of  $Z$ .

We also fix  $x, y \in A$  satisfying the following conditions:

- The discriminant locus  $Z_U$  of  $U \rightarrow T$  is the closed subscheme corresponding to  $(x, y)$ .
- $xy = \pi \in R$ .
- The corresponding map  $R[x, y]/(xy - \pi) \rightarrow A$  induces an isomorphism when both sides are completed with respect to the ideal  $(x, y)$ .

**Remark 5.39.** The local deformation theory of stable curves guarantees the existence of such  $x, y$  possibly after some étale base change. For instance, it follows immediately from [Stacks, Tag 0CBY].

**Remark 5.40.** Jarvis [Jar98] calls such  $x, y \in A$  *local coordinates*.

Now we are ready to prove the main lemma of this section.

**Lemma 5.41.**  *$F \rightarrow T$  is a closed immersion.*

*Proof.* We pick local coordinates  $x, y \in A$  around a preferred neighbourhood  $U = \operatorname{Spec} A \rightarrow C$  around the discriminant locus.

Denote by  $Z$  the discriminant locus of  $U \rightarrow T$  which is carved out by  $J = (x, y)$  and  $xy = \pi \in R$ . By definition, completion of  $A$  with respect to  $J$  satisfies  $\hat{A}_J \xrightarrow{\sim} \hat{R}_{(\pi)}[[x, y]]/(xy - \pi)$ . Let  $\hat{Z} := \operatorname{Spec} \hat{A}_J$  and note that the map  $(U \setminus Z) \sqcup \hat{Z} \rightarrow U$  is an fpqc cover.

Let  $\psi: (E(p_1, p_1)^2, s) \rightarrow (E(p_2, p_2)^2, s)$  be an isomorphism induced by the pre-sync data. Consider the morphism  $\psi': \operatorname{Sym}^2 \mathcal{E}_1^2 \rightarrow \mathcal{E}_2^4$  induced from  $\psi$ . Denote the standard generators by  $\xi_1, \xi_2 \in E(p_1, p_1)$  and  $\zeta_1, \zeta_2 \in E(p_2, p_2)$ .

Recall from Lemma 5.22 that  $\psi$  is a sync data iff  $\psi'(\xi_1^2 \xi_2^2 - (\xi_1 \xi_2)^2) = 0$ . This is always satisfied on  $U \setminus Z$ , so we concentrate on  $\hat{Z}$ .

Using standard arguments (§A2.3 [Eis95]) one concludes that:

$$E(p, p)^2 = \operatorname{coker} \left( L(p, p): \hat{A}_J^{\oplus 4} \rightarrow \hat{A}_J^{\oplus 3} \right)$$

where

$$L(p, p) = \begin{pmatrix} y & -p & 0 & 0 \\ -p & x & y & -p \\ 0 & 0 & -p & x \end{pmatrix}. \quad (5.3.2)$$

We want can lift  $\psi$  to a map  $\hat{A}_J^{\oplus 3} \rightarrow \hat{A}_J^{\oplus 3}$ . Using the 4 relations above we construct a lift such that the corresponding  $3 \times 3$  matrix contains no terms involving  $y$  in the first row,  $x$  or  $y$  in the second row,  $x$  in the third row.

Such a lift is unique and will be denoted by  $[\psi]$ . A direct calculation shows that  $\psi$  commutes with the two  $s$ 's iff the lift  $[\psi]$  satisfies the following equality

$$[\psi] = \begin{pmatrix} 1 & 0 & 0 \\ a_1 & u & a_2 \\ 0 & 0 & 1 \end{pmatrix} \quad (5.3.3)$$

where  $a_1, a_2 \in \operatorname{Ann}(p_2) = \operatorname{Ann}(p_1)$  and  $u \in R^\times$  is such that  $up_2 = p_1$ .

In terms of the entries of this matrix we have the following equality:

$$\psi'(\xi_1^2 \xi_2^2 - (\xi_1 \xi_2)^2) = \zeta_1^2 \zeta_2^2 + a_1 \zeta_2^2 (\zeta_1 \zeta_2) + a_2 \zeta_1^2 (\zeta_1 \zeta_2) + (a_1 a_2 - u^2) (\zeta_1 \zeta_2)^2.$$

As before we can calculate a presentation of  $E(p_2, p_2)^4$ . This presentation looks similar to Equation 5.3.2 but with an extra block. This shows that  $\psi$  is a sync data iff  $a_1 = a_2 = 0$  and  $u^2 = 1$ . Since we assume  $\operatorname{Spec} R$  is connected, this forces  $u = \pm 1$ .

In fact,  $u$  can attain only one of these values. To determine which, we can pullback our representation  $[\psi]$  to  $t$ . By the previous paragraph, and our hypothesis on  $t$ , there is an  $\varepsilon \in \{\pm 1\}$  such that:

$$[\psi_t] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define an ideal  $I \subset R$  so that  $\hat{I}_{(\pi)} \subset \hat{R}_{(\pi)}$  equals  $(a_1, u - \varepsilon, a_2)$  and  $I|_{D(\pi)} = 0$  on the principle open subset  $D(\pi) \subset T$ .

Along any  $T' \rightarrow T$  let  $Z'$  be the discriminant locus of  $U' = U \times_T T' \rightarrow T'$ . This gives us a commutative map:

$$\begin{array}{ccc} \hat{Z}' & \longrightarrow & \hat{Z} \\ \downarrow & & \downarrow \\ U' & \longrightarrow & U \end{array}$$

The matrix entries of  $[\psi]$  pullback as expected from  $Z$  to  $Z'$ . Therefore  $\psi_{U'}$  satisfies the sync condition iff the map  $T' \rightarrow T$  sends the three terms  $a_1, a_2, u - \varepsilon$  to 0. Therefore,  $F$  is represented by  $\text{Spec } R/I$ .  $\square$

At this point we proved Proposition 5.35. However, we make one final deduction from the contents of this section.

**Set-up 5.42.** Assume  $R$  to be a complete local noetherian ring with an algebraically closed residue field.

**Lemma 5.43.** *With  $R$  as in Set-up 5.42 and  $C \rightarrow \text{Spec } R$  a stable curve, the sync condition can be checked in the formal neighbourhood of the discriminant locus.*

*Proof.* This follows from the proof of the previous result, with the additional observation that no étale base change is required.  $\square$

**Corollary 5.44.** *When the base ring is as in Set-up 5.42, the two definitions of a multiple root given by Definition 3.4 and Definition 5.21 agree.*

*Proof.* Use the lemma above to reduce the problem to the formal neighbourhood of the discriminant locus. Lemma 2.22 describes the isomorphisms explicitly that are allowed by Definition 3.4. On the other hand, in the proof of Lemma 5.41 we gave explicit conditions for the sync condition. The two descriptions agree.  $\square$

### 5.3.3 The diagonal

**Remark 5.45.** Now that we proved in Theorem 5.29 that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is algebraic, it follows that the relative diagonal  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{M}$  is representable by algebraic spaces.

**Proposition 5.46.** *The diagonal  $\Delta : \overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$  is finite and unramified.*

*Proof.* Fix a morphism  $B \rightarrow \overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$  where  $B$  is a scheme. Since the stacks under consideration are locally noetherian and the question is local on the target of  $\Delta$  we may assume  $B$  to be noetherian.

This morphism defines a curve over  $B$  and a pair of multiple roots  $\mathfrak{R} = (\mathcal{R}, \Phi)$  and  $\mathfrak{R}' = (\mathcal{R}', \Phi')$ . We want to show that the functor  $\text{Iso}(\mathfrak{R}, \mathfrak{R}')$  is represented by a finite and unramified scheme over  $B$ . In light of Remark 5.45 the map  $\Delta$  is representable by algebraic spaces.

An isomorphism of a multiple root is an isomorphism of the underlying sequence of roots compatible with the sync data. Therefore, we have a map  $\text{Iso}(\mathfrak{R}, \mathfrak{R}') \rightarrow \text{Iso}(\mathcal{R}, \mathcal{R}') = \prod_{i=1}^m \text{Iso}(\mathcal{R}_i, \mathcal{R}'_i)$ , where the product of the functors  $\text{Iso}(\mathcal{R}_i, \mathcal{R}'_i)$  is to be taken over  $B$ .

Each of the isomorphism functors  $\text{Iso}(\mathcal{R}_i, \mathcal{R}'_i)$  is represented by a finite and unramified scheme over  $B$  as shown in §4.1.4.3 of [Jar98]. Hence, their product over  $B$  is also finite and unramified over  $B$ . We will now show that  $\text{Iso}(\mathfrak{R}, \mathfrak{R}')$  is a connected component of  $\text{Iso}(\mathcal{R}, \mathcal{R}')$ , and hence finite and unramified over  $B$ .

To show that  $\text{Iso}(\mathfrak{R}, \mathfrak{R}')$  is a component of  $\text{Iso}(\mathcal{R}, \mathcal{R}')$ , we will prove that the property of being compatible with the sync data for a sequence of isomorphisms both specializes and generalizes. Since we now know that the diagonal is representable this will conclude the proof. Furthermore, since  $\text{Iso}(\mathcal{R}, \mathcal{R}')$  is locally noetherian we need only use *discrete* valuation rings.

Let  $R$  be a complete DVR with generic point  $\eta$  and special point  $\sigma$ . Consider a map  $\text{Spec } R \rightarrow \text{Iso}(\mathcal{R}, \mathcal{R}')$ , which gives us a family of curves  $\mathcal{X} \rightarrow \text{Spec } R$  and a sequence of isomorphism  $(\varphi_i: (\mathcal{E}_i, b_i) \rightarrow (\mathcal{E}'_i, b'_i))_{i=1}^m$  between the roots.

We can focus on one node at a time because distinct nodes do not interact with one another. In light of Lemma 5.44, we can use the simpler definition of a multiple root defined for complete noetherian local rings. But with this definition, compatibility with the sync data has to be checked for each pair of indices  $i, j$  separately. Therefore, we can assume  $m = 2$  to simplify notation.

Pick a component of the discriminant locus. We need only focus our attention to the formal completion of this locus. Recall that in the formal neighbourhood of a node the isomorphisms  $\varphi_i$  between  $\mathcal{E}_i$  and  $\mathcal{E}'_i$  are of the form  $\begin{bmatrix} \varepsilon'_i & 0 \\ 0 & \varepsilon''_i \end{bmatrix}$ , with  $\varepsilon'_i, \varepsilon''_i \in \{\pm 1\}$  (this is Lemma 2.22). The symmetric square  $\varphi_i^2$  equals  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_i & 0 \\ 0 & 0 & 1 \end{bmatrix}$ , where  $\varepsilon_i = \varepsilon'_i \varepsilon''_i$ .

In the meantime, the sync condition on either side is represented by an isomorphism  $\psi: \mathcal{E}_1^2 \rightarrow \mathcal{E}_2^2$  on the first pair of roots and  $\psi'$  on the second pair of roots. We know that  $\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $\psi' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon' & 0 \\ 0 & 0 & 1 \end{bmatrix}$  where  $\varepsilon, \varepsilon' \in \{\pm 1\}$ .

These four isomorphisms give a commuting diagram iff  $\varepsilon \varepsilon_1 = \varepsilon' \varepsilon_2$ . Therefore we have reduced the argument to showing that this equality holds over the special fiber iff it holds over the generic fiber.

First and foremost, if the component of the discriminant locus we are studying is an isolated node, then  $\pi \in R$  representing this node is not zero. In particular, this forces that there be precisely one isomorphism between the squares of these roots (see Remark 2.24). Therefore the desired equality holds by uniqueness.

On the other hand, if the node is defined over the generic fiber then it persists over the entire base. In this case, each of the signs  $\varepsilon_i$  are constant throughout this discriminant locus. Therefore, the equality  $\varepsilon \varepsilon_1 = \varepsilon' \varepsilon_2$  holds iff it holds at any one part of this locus.  $\square$

**Corollary 5.47.**  $\overline{\mathcal{S}}^m(\mathcal{N})$  is a Deligne–Mumford stack.

*Proof.* The proposition above combined with the fact that  $\mathcal{M}$  is a DM stack yields this result.  $\square$

## 5.4 Additional properties

### 5.4.1 Smoothness

For half of this paper we studied the local structure of our stacks  $\overline{\mathcal{S}}^m(\mathcal{N})$ . The following is a useful summary of what we know.



**Theorem 5.48.** *If  $\mathcal{M} \rightarrow \overline{\mathcal{M}}_g$  is smooth then  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow S$  is smooth.*

*Proof.* Smoothness can be checked in geometric formal neighbourhoods of points. But smoothness at geometric formal neighbourhoods is precisely the content of Corollary 4.24.  $\square$

**Remark 5.49.** Conceptually, we are not just proving that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is smooth but also that our definition of multiple roots is the “right” one. Since we can now use intersection theory on the stacks  $\overline{\mathcal{S}}^m(\mathcal{N})$  to solve enumerative problems.

#### 5.4.2 A proper compactification

We now prove that  $\overline{\mathcal{S}}^m(\mathcal{N})$  is indeed a ‘closure’ of  $\mathcal{S}^m(\mathcal{N})$  when we would expect.

**Lemma 5.50.** *If  $\mathcal{C} \rightarrow \mathcal{M}$  is generically smooth then  $\mathcal{S}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}^m(\mathcal{N})$  is a dense open immersion.*

*Proof.* Since all roots on a smooth curve are locally free, it suffices to show that any root can be deformed onto a smooth curve. Provided that any singular curve  $X$  can be deformed to a smooth curve over  $\mathcal{M}$ , it follows immediately from the local deformation functors discussed in Section 4 that any tuple of roots on  $X$  can also be deformed onto a smooth curve over  $\mathcal{M}$ . In particular, this is immediate from Theorem 4.22.  $\square$

**Lemma 5.51.** *The morphism  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{M}$  is proper.*

*Proof.* We prove this by the valuative criterion of properness and induction on  $m \geq 1$ . The result for  $m = 1$  is part of Proposition 5.3. Then we need only show that the map  $\mathcal{Y} := \overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \overline{\mathcal{S}}^{m-1}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  defined in Remark 5.27 is proper. Clearly the diagonal is locally noetherian so we may restrict to checking the valuative criterion using complete DVRs.

Let  $R$  be a complete discrete valuation ring, with residue field  $K$ . Consider a 2-commutative diagram:

$$\begin{array}{ccc} \mathrm{Spec} K & \longrightarrow & \overline{\mathcal{S}}^m(\mathcal{N}) \\ \downarrow & \searrow & \downarrow \\ \mathrm{Spec} R & \longrightarrow & \mathcal{M} \end{array}$$

This means that we have a synchronized  $(m-1)$ -tuple of roots and an  $m$ -th root over the curve  $C_R \rightarrow \mathrm{Spec} R$ . Furthermore, these roots are all synchronized over the general fiber. However, as we demonstrated in the proof of Proposition 5.46 a synchronization on the generic fiber over a complete DVR extends to the entire family uniquely.  $\square$

**Remark 5.52.** If  $\mathcal{C} \rightarrow \mathcal{M}$  is not assumed to be generically smooth the result will certainly not hold, even when  $m = 1$ . For example one could take  $\mathcal{M} = \mathrm{Spec} k$ . In this case, the isomorphism classes of roots of a fixed line bundle form a discrete set so one can not deform the locally free roots on to the non-free roots.

### 5.4.3 Coarse moduli space

**Lemma 5.53.**  $\overline{\mathcal{S}}^m(\mathcal{N}) \rightarrow \mathcal{M}$  is quasi-finite.

*Proof.* We will use the fact that  $\overline{\mathcal{S}}(\mathcal{N}) \rightarrow \mathcal{M}$  is quasi-finite 5.9. Fix a geometric point of the  $m$ -fold product  $\overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ . Our goal is to show that there are finitely many synchronizations on the corresponding sequence of roots.

In light of Corollary 5.44 we may use Definition 3.4. But then we may reduce to the case  $m = 2$ . In the formal neighbourhood of each node where both roots are singular, we need to show that the number of isomorphism between the two roots is finite. This is implied by Lemma 2.22 which says that there are precisely 4 such isomorphisms.  $\square$

**Remark 5.54.** The proof below is a slight adaptation of Proposition 3.1.1 [Jar00].

**Proposition 5.55.** *If the coarse moduli space of  $\mathcal{M}$  is projective over  $S$  then so is the coarse moduli space of  $\overline{\mathcal{S}}^m(\mathcal{N})$  projective over  $S$ .*

*Proof.* It is well known that separated Deligne–Mumford stacks are coarsely represented by algebraic spaces (e.g. Corollary 1.3.1 [KM97]). So we let  $X = \text{coarse}(\overline{\mathcal{S}}^m(\mathcal{N}))$  and  $Y = \text{coarse}(\mathcal{M})$  be these coarse moduli spaces with  $f: X \rightarrow Y$  the natural map between them.

This map  $f$  is proper because the corresponding map between the stacks is proper. Also  $f$  is quasi-finite by Lemma 5.53. Therefore  $f$  is finite and hence projective. When  $Y \rightarrow S$  is projective then so is  $X \xrightarrow{f} Y \rightarrow S$ .  $\square$

## A Geometric interpretation

The main purpose of this appendix is to explain how torsion-free sheaves of rank-1 and blow-ups of curves are really the same thing. We then recall how limit roots, as defined in [CCC07], agree with the (torsion-free) roots of [Jar98]. This is done in order to explain the underlying geometric intuition behind our definition for multiple roots.

### A.1 Blow-ups of curves

Let  $k$  be an algebraically closed field with  $\text{char } k \neq 2$ . Let  $C$  be a stable curve over  $k$  and  $N$  a line bundle on  $C$  with  $\deg N$  divisible by 2.

**Remark A.1.** We don't need such strong hypotheses on  $k$ . Most of what we say in this appendix will work if the word *node* is replaced with *split node*. The rest will work if  $\text{char } k \neq 2$  and  $\sqrt{k} = k$ . But we will not use this fact.

Suppose  $x_1, \dots, x_v \in C$  are some of the nodes of  $C$ . Let  $I := I(\mathfrak{x}) \subset \mathcal{O}_C$  be the ideal sheaf corresponding to the subscheme  $\mathfrak{x} := \{x_1, \dots, x_v\} \subset C$ . Define  $\pi: X = \text{Proj}_C(\text{Sym}^* I) \rightarrow C$ . Observe that  $\pi$  is an isomorphism in the open set  $C \setminus \mathfrak{x}$ . And for any  $x \in \mathfrak{x}$ , the fiber of  $X \rightarrow C$  over  $x$  is isomorphic to  $\mathbb{P}_k^1$ . This motivates us in making the following abuse of notation:

**Definition A.2.** For a subset of the nodes  $\mathfrak{x} \subset C$ , the corresponding construction  $\pi: X \rightarrow C$  will be called a *blow-up* of  $C$  at  $\mathfrak{x}$ . Denote this object by  $\pi: \text{Bl}_{\mathfrak{x}} C \rightarrow C$  and allow for  $\mathfrak{x} = \emptyset$ . If  $X$  is the blow-up of a stable curve then  $X$  is called a *quasi-stable curve*.

**Remark A.3.** This is indeed an abuse of notation. The actual blow-up of  $C$  at  $\mathfrak{x}$  is defined via the Rees algebra and not the symmetric algebra of  $I(\mathfrak{x})$ . So the actual blow-up construction partially normalizes  $C$  at  $\mathfrak{x}$ . The canonical surjective map  $\text{Sym}^*(I) \rightarrow \text{Rees}(I)$  induces a closed immersion from the partial normalization of  $C$  at  $\mathfrak{x}$  into the blow-up.

**Remark A.4.** The Proj construction yields, in addition, a line bundle on the total space. Only with this line bundle is this object truly meaningful, i.e., it satisfies a universal property (see [Stacks, Tag 01NS]). Indeed if we were to consider blow-ups together with their defining line bundles, we would go full circle and recognize that these objects are in correspondence with torsion-free sheaves of rank-1.

**Definition A.5.** Given any  $Y \rightarrow C$  if there is a subset  $\mathfrak{x} \subset C$  of the nodes and an isomorphism

$$\begin{array}{ccc} Y & \xrightarrow{\sim} & \text{Bl}_{\mathfrak{x}} C \\ & \searrow & \swarrow \\ & C & \end{array}$$

then we will also call  $Y \rightarrow C$  a blow-up of  $C$  at  $\mathfrak{x}$ .

**Definition A.6.** Let  $\pi: X \rightarrow C$  be a blow-up of the nodes  $\mathfrak{x} \subset C$ . Then for each  $x \in \mathfrak{x}$  we will call the fiber  $\pi^{-1}(x) \subset X$  an *exceptional component* of  $X$ .

## A.2 Limit roots on curves

We now recall the notion of a limit root given in Definition 2.1.1 of [CCC07].

**Definition A.7.** Consider a triplet  $(\pi: X \rightarrow C, L, \alpha: L^{\otimes 2} \rightarrow \pi^* N)$  where  $\pi$  is a blowup and  $L$  is a line bundle on  $X$  of degree  $\frac{\deg N}{2}$ . This triple is called a *limit root* of  $N$  if the following are satisfied:

- $L$  has degree 1 on each exceptional divisor of  $\pi$ .
- $\alpha$  is an isomorphism in the complement of the exceptional components of  $X$ .

**Remark A.8.** We get to omit condition (iii) in Definition 2.1.1 of [CCC07] because we specified the degree of  $L$  and because we are considering *square* roots.

## A.3 Families of limit roots

Let  $\mathcal{C} \rightarrow T$  be a family of stable curves over a scheme  $T$ , where  $T$  is defined over  $\mathbb{Z}[\frac{1}{2}]$ . Let  $\mathcal{N}$  be a line bundle on  $\mathcal{C}$  of relative degree  $d$ , which we assume to be even. The following three definitions are from [CCC07].

**Definition A.9.** Suppose that  $\pi: \mathcal{X} \rightarrow \mathcal{C}$  is a morphism such that,  $\mathcal{X} \rightarrow T$  is a family of nodal curves and for each geometric point  $t \rightarrow T$  the fiber  $\pi_t: \mathcal{X}_t \rightarrow \mathcal{C}_t$  is a blowup in the sense of Definition A.2. Then we will call  $\pi: \mathcal{X} \rightarrow \mathcal{C}$  a *family of blow-ups*.

**Definition A.10.** Let  $\pi: \mathcal{X} \rightarrow \mathcal{C}$  be a family of blowups,  $\mathcal{L}$  a line bundle on  $\mathcal{X}$  and  $\alpha: \mathcal{L}^{\otimes 2} \rightarrow \pi^*\mathcal{N}$  a morphism. If  $(\mathcal{X} \xrightarrow{\pi} \mathcal{C}, \mathcal{L}, \alpha)$  restricts on each geometric fiber to a limit root as in Definition A.7, then we will call  $(\mathcal{X} \xrightarrow{\pi} \mathcal{C}, \mathcal{L}, \alpha)$  a *family of limit roots*.

**Definition A.11.** For  $i = 1, 2$  let  $(\pi_i: \mathcal{X}_i \rightarrow \mathcal{C}, \mathcal{L}_i, \alpha_i)$  be two families of limit roots. An isomorphism between them is a pair  $(f, g)$  where  $f: \mathcal{X}_1 \rightarrow \mathcal{X}_2$  is an isomorphism over  $\mathcal{C}$  and  $g: \mathcal{L}_1 \rightarrow f^*\mathcal{L}_2$  is an isomorphism such that  $\alpha_1 = f^*\alpha_2 \circ g^{\otimes 2}$ .

We can now define the moduli space corresponding to limit roots.

**Definition A.12.** Let  $\overline{\mathcal{S}}(\mathcal{N})' \rightarrow T$  be the category fibered in groupoids, associating to each  $T' \rightarrow T$  the groupoid of families of limit roots of  $\mathcal{N}|_{T'}$  over  $\mathcal{C}|_{T'} \rightarrow T'$ .

### A.3.1 Relation to torsion-free roots

Suppose  $(\mathcal{E}, b)$  is a root of  $\mathcal{N}$  on  $\mathcal{C}$ . Define  $\mathbb{P}(\mathcal{E}) := \underline{\text{Proj}}_{\mathcal{C}}(\text{Sym}^* \mathcal{E})$  with  $\pi: \mathbb{P}(\mathcal{E}) \rightarrow \mathcal{C}$  the structure map and  $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$  the line bundle corresponding to the Proj construction. Notice that  $\pi$  is a family of blow-ups.

There are natural surjective maps  $\pi^*\mathcal{E}^d \rightarrow \mathcal{L}^{\otimes d} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$  for each  $d \geq 0$  and there is the map  $\pi^*b: \pi^*\mathcal{E}^2 \rightarrow \pi^*\mathcal{N}$ . As is shown in §3.1.3 of [Jar98] there is a natural map  $\alpha$  making the following diagram commute:

$$\begin{array}{ccc} \pi^*\mathcal{E}^2 & & \\ \downarrow & \searrow \pi^*b & \\ \mathcal{L}^2 & \xrightarrow{\alpha} & \pi^*\mathcal{N} \end{array}$$

**Proposition A.13.** Let  $(\mathcal{E}, b)$  be a root of  $\mathcal{N}$ . Then  $(\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathcal{C}, \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1), \alpha)$ , constructed above, is a family of limit roots of  $\mathcal{N}$ .

*Proof.* Both Proj and Sym constructions behave well with respect to base change. So we may reduce to  $T = \text{Spec } k$  where  $k$  is an algebraically closed field.

Let  $L = \mathcal{O}(1)$  and note  $\pi_*L \simeq \mathcal{E}$ , see Lemma 3.1.4.(2) [Jar98]. To see that  $L$  has degree one over any exceptional fiber  $E$  over a node  $x$  we simply observe  $h^0(E, L|_E) = \dim_k \mathcal{E}|_x = 2$ . Since  $E \simeq \mathbb{P}_k^1$  we are done.

The map  $\alpha$  is an isomorphism away from the exceptional divisors because  $b$  is an isomorphism away from the corresponding nodes. The degrees of  $L$  and  $\mathcal{E}$  agree because  $\pi_*L \simeq \mathcal{E}$ . This completes the proof.  $\square$

Conversely, given a family of limit roots  $(\pi: \mathcal{X} \rightarrow \mathcal{C}, \mathcal{L}, \alpha)$ , let  $\mathcal{E} := \pi_*\mathcal{L}$ . Then, using Lemma 3.1.4.(2) [Jar98] again, we have  $\pi_*\mathcal{L}^2 \simeq \mathcal{E}^2$ . Using the adjunction map  $a: \pi_*\pi^*\mathcal{N} \rightarrow \mathcal{N}$  we may define  $b := a \circ \pi_*\alpha: \mathcal{E}^2 \rightarrow \pi_*\pi^*\mathcal{N} \rightarrow \mathcal{N}$ .

**Proposition A.14.** The tuple  $(\mathcal{E}, b)$  obtained in this way is a root of  $\mathcal{N}$

*Proof.* This is similar to the proposition above. The main ingredients are Proposition 3.1.2.(3) and Proposition 3.1.5 of [Jar98] which says that  $\pi_*\mathcal{L}$  is torsion-free and  $b$  is of the right form respectively.  $\square$

In summary, we conclude that families of limit roots (using quasi-stable curves) and families of roots (using torsion-free sheaves) are in fact equivalent notions. More precisely we may state:

**Corollary A.15.** *The categories  $\overline{\mathcal{S}}(\mathcal{N})'$  and  $\overline{\mathcal{S}}(\mathcal{N})$  are equivalent.*

*Proof.* The two propositions above constructs the functors which are clearly quasi-inverses to one another. The fact that these constructions behave functorially follows from the functorial behavior of  $\underline{\text{Proj}}$  and pushforward.  $\square$

## A.4 Multiple limit roots

Let  $\mathcal{C} \rightarrow T$  be a stable curve. We want to consider  $m$ -tuples of limit roots of  $\mathcal{N}$  over  $\mathcal{C}$ . Before we give our definition, let us explore some of the difficulties and dead ends.

### A.4.1 Motivation

It will be sufficient to assume  $m = 2$ . Let us consider  $\mathcal{Y} := \overline{\mathcal{S}}(\mathcal{N})' \times_T \overline{\mathcal{S}}(\mathcal{N})'$  for a moment. If this moduli space were to work, we could have stopped right here. But the main problem with  $\mathcal{Y}$  can already be seen at its geometric points.

Let  $t = \text{Spec } k \rightarrow T$  be a geometric point. Let  $C = \mathcal{C}|_t$  and  $N = \mathcal{N}|_t$ . If  $\text{Spec } k \xrightarrow{y} \mathcal{Y}$  is a geometric point lying over  $t$ , then  $y$  corresponds to two limit roots of  $N$  over  $C$ . We will denote these by  $(X_i \xrightarrow{\pi_i} C, L_i, \alpha_i)$  for  $i = 1, 2$ .

Let us consider a particularly simple case. Suppose that  $\pi_i$ 's are both blow-ups of a single node  $x \in C$ . Then the two curves  $X_1$  and  $X_2$  are isomorphic, but not canonically: The exceptional divisors  $E_i := \pi_i^{-1}(x)$  are not uniquely identified with one another. There is a  $k^*$ -torsor of such isomorphisms between the  $X_i$ 's.

We could ask for an isomorphism  $f: X_1 \xrightarrow{\sim} X_2$  over  $C$  such that  $f^* L_2^{\otimes 2} \simeq L_1^{\otimes 2}$  over  $\pi_1^* \mathcal{N}$ . This restriction is natural as we are asking for the two limit roots to square to the same line bundle. Moreover, there are now only two possible isomorphisms between the  $X_i$ 's. Nevertheless, this is still not unique.

This demonstrates the main shortcoming of the fiber product  $\mathcal{Y} = \overline{\mathcal{S}}(\mathcal{N})' \times_T \overline{\mathcal{S}}(\mathcal{N})'$ . The tuples of limit roots can not be viewed as line bundles on a single family of curves. Hence the objects parametrized by  $\mathcal{Y}$  are rather awkward. We summarize this here.

**Proposition A.16.** *Objects of  $\overline{\mathcal{S}}^m(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}^m(\mathcal{N})$  are not geometrically meaningful.*

*Proof.* Recall  $\overline{\mathcal{S}}^m(\mathcal{N}) \simeq \overline{\mathcal{S}}(\mathcal{N})'$  so that we reduce to the argument above. Objects in  $\mathcal{Y}$  require two different curves and line bundles on them, this is not when wanting to compare the line bundles.  $\square$

The following fix to our problem readily suggests itself. If  $X_i$  is a blow-up of  $\mathfrak{x}_i \subset C$  then let us fix a blow-up  $\pi: X \rightarrow C$  of  $\mathfrak{x} = \mathfrak{x}_1 \cup \mathfrak{x}_2$ . The blow-up  $\pi$  non-canonically factors through  $\pi_i$ 's so we must incorporate this factorization into our data to remove any ambiguity. In other words, we want to fix  $\rho_i: X \rightarrow X_i$  such that  $\pi = \pi_i \circ \rho_i$ . If this data is available,  $L_i$ 's can be canonically pulled back to  $X$  so that they share the same curve.

There is one fatal flaw with this approach however. Namely that there are way too many  $\rho_i$ 's to give a good moduli space. Recall that when  $X_i$ 's were the blow-ups of a single point  $x \in C$ , the tuples  $(\rho_1, \rho_2)$  up to isomorphism would form a  $k^*$ -torsor.

When  $X_1$  and  $X_2$  are isomorphic we can fix this flaw, as before, by requiring that  $\rho_1^* L_1^{\otimes 2} \simeq \rho_2^* L_2^{\otimes 2}$  over  $\pi^* N$ . When this is not the case, we have to make precise the statement that  $\rho_i^* L_i^{\otimes 2}$ 's are isomorphic *in the locus where this makes sense*.

In order to motivate our definition, let us make one final observation. Given  $\rho_i: X \rightarrow X_i$  and  $\rho_i^* L_i$ 's on  $X$  we can forget about the partial blow-ups  $\pi_i: X_i \rightarrow C$ . Blowing down exceptional components of  $X$  on which  $\rho_i^* L_i$  has degree 0 would recover  $X_i \rightarrow C$ . This procedure can also be done in families of limit roots.

#### A.4.2 Multiple limit roots

Assume  $m \geq 1$  once again.

**Definition A.17.** Let  $\pi: \mathcal{X} \rightarrow \mathcal{C}$  be a family of blow-ups,  $\mathcal{L}$  a line bundle on  $\mathcal{X}$  and  $\alpha: \mathcal{L}^{\otimes 2} \rightarrow \pi^* \mathcal{N}$  a morphism. If there is a limit root  $(\pi': \mathcal{X}' \rightarrow \mathcal{C}, \mathcal{L}', \alpha')$  and a factorization  $\pi = \pi' \circ \rho$  such that  $(\mathcal{L}, \alpha)$  is the pullback of  $(\mathcal{L}', \alpha')$  under  $\rho$  then we will say that  $(\mathcal{L}, \alpha)$  *stabilizes to a limit root* and the map  $\rho: \mathcal{X} \rightarrow \mathcal{X}'$  is the *partial stabilization with respect to  $\mathcal{L}$* .

**Remark A.18.** Partial stabilization simply contracts the unstable components of each fiber on which  $\mathcal{L}$  has degree 0. See also Lemma 5.12 to see how this condition behaves.

**Definition A.19.** Let  $\pi: \mathcal{X} \rightarrow \mathcal{C}$  be a family of blow-ups. Let  $\mathfrak{L} := \{\mathcal{L}_i, \alpha_i: \mathcal{L}_i^{\otimes 2} \rightarrow \pi^* \mathcal{N}\}_{i=1}^m$  be such that each  $(\mathcal{L}_i, \alpha_i)$  stabilizes to a limit root. Consider a line bundle  $\mathcal{L}$  and a sequence of morphisms  $\varphi_i: \mathcal{L} \rightarrow \mathcal{L}_i^{\otimes 2}$  satisfying the following:

- $\alpha_i \circ \varphi_i = \alpha_j \circ \varphi_j$  for each  $i, j$ .
- Each  $\varphi_i$  restricts to an isomorphism on  $V_i$ .

Then, we will call  $\mathfrak{F} = (\varphi_i)_{i=1}^m$  a *synchronization data*. The tuple  $(\pi, \mathfrak{L}, \mathfrak{F})$  will be called a *multiple limit root*. An isomorphism of multiple limit roots is an isomorphism of the limit roots commuting with the synchronization data.

**Definition A.20.** Let  $\bar{\mathcal{S}}^m(\mathcal{N})' \rightarrow T$  be the stack associating to  $T' \rightarrow T$  the groupoid of multiple limit roots of  $\mathcal{N}|_{T'}$  over  $\mathcal{C}|_{T'} \rightarrow T'$ .

#### A.4.3 Relation to torsion-free roots

It remains to show that multiple limit roots (Definition A.19) and multiple roots (Definition 5.15) are in fact equivalent. More precisely, we prove the following.

**Proposition A.21.** *The two categories  $\bar{\mathcal{S}}^m(\mathcal{N})'$  and  $\bar{\mathcal{S}}^m(\mathcal{N})$  are equivalent.*

*Proof.* Given a multiple limit root  $(\pi, \mathfrak{L}, \mathfrak{F})$  we can push forward each limit root to obtain a root as in Section A.3.1. Denote these roots by  $\mathfrak{R} = (\mathcal{E}_i, b_i)_{i=1}^m$ . Let  $D = \bigoplus_{d \geq 0} \pi_* \mathcal{O}_{\mathcal{X}}(d)$  and define  $\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i$  using  $\varphi_i$ . Let  $\Psi = (\psi_i)_{i=1}^m$ . It is easy to check that  $(\mathfrak{R}, \Psi)$  is a multiple root.

Conversely, suppose we are given a multiple root  $(\mathfrak{R}, \Phi)$  of  $\mathcal{N}$ . We will now construct a multiple limit root as follows.

With  $\mathfrak{R} = (\mathcal{E}_i, b_i)_{i=1}^m$  consider  $\pi_i: \mathcal{X}_i \rightarrow \mathcal{C}$  where  $\mathcal{X}_i = \underline{\text{Proj}}_{\mathcal{C}}(\text{Sym}^* \mathcal{E}_i)$ . Define  $\mathcal{L}_i := \mathcal{O}_{\mathcal{X}_i}(1)$  and  $\alpha_i: \mathcal{L}_i^{\otimes 2} \rightarrow \pi_i^* \mathcal{N}$  as in Section A.3.1.

The synchronization data  $\Psi = (\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i)_{i=1}^m$  is such that  $D$  is a graded sheaf of algebras and the morphisms  $\psi_i: D \rightarrow \text{Sym}^{2*} \mathcal{E}_i$ , which are compatible with  $b_i$ 's, restrict to isomorphisms on  $V_i$ 's.

Let  $\pi: \mathcal{X} = \underline{\text{Proj}}_{\mathcal{C}} D \rightarrow \mathcal{C}$  with  $\mathcal{F} := \mathcal{O}_{\mathcal{X}}(1)$ . Over  $V_i$  where  $\psi_i$  is an isomorphism, we get an isomorphism  $\rho_{V_i}: \mathcal{X}|_{V_i} \rightarrow \mathcal{X}_i|_{V_i}$ . Since  $\mathcal{X}_i$  is isomorphic to  $\mathcal{C}$  away from the discriminant loci contained in  $V_i$ , we can extend  $\rho_{V_i}$  to  $\rho_i: \mathcal{X} \rightarrow \mathcal{X}_i$  satisfying  $\pi = \pi_i \circ \rho_i$ . The morphisms  $\psi_i$  yields another morphism  $\varphi_i: \mathcal{O}_{\mathcal{X}}(1) \rightarrow \rho_i^* \mathcal{O}_{\mathcal{X}_i}(2)$  which restricts to an isomorphism over  $V_i$ .

Therefore, if we set  $\mathcal{L} := (\rho_i^* \mathcal{O}_{\mathcal{X}_i}(1), \rho_i^* \alpha_i)_{i=1}^m$  and  $\mathfrak{F} = (\varphi_i: \mathcal{O}_{\mathcal{X}}(1) \rightarrow \mathcal{O}_{\mathcal{X}_i}(2))_{i=1}^m$  then the tuple  $(\pi, \mathcal{L}, \mathfrak{F})$  is a multiple limit root.

These two constructions are functorial because pushforward and  $\underline{\text{Proj}}$  are functorial.  $\square$

## B Non-normal product space

We claimed in the introduction that the  $m$ -fold product  $\overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$  is non-normal in general. We will prove this here.

For the purposes of demonstration, it will be sufficient to assume  $m = 2$ ,  $\mathcal{M} = \overline{\mathcal{M}}_g$  the moduli space of stable curves of genus  $g$  with  $\mathcal{C} \rightarrow \mathcal{M}$  the universal curve over it. Take  $\mathcal{N} = \omega_{\mathcal{C}/\mathcal{M}}$  to be the relative dualizing sheaf.

Let  $\mathcal{Y} = \overline{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \overline{\mathcal{S}}(\mathcal{N})$ . For  $k$  an algebraically closed field, pick a general irreducible curve  $C$  over  $k$  having a single node  $x \in C$ . This corresponds to a point  $c: \text{Spec } k \rightarrow \mathcal{M}$ . Let  $(\mathcal{E}_i, b_i)$  for  $i = 1, 2$  be non-isomorphic roots of  $\omega_{C/k}$  such that  $\mathcal{E}_i$ 's are both non-free at the node  $x$ . This corresponds to a point  $y \rightarrow \mathcal{Y}$  lying over  $c$ .

**Proposition B.1.** *The moduli stack  $\mathcal{Y}$  is non-normal at  $y$ .*

*Proof.* Using Theorem 4.22 for each of the roots separately, and arguing as in Section 3.2 we conclude that the local deformation functor at  $y$  of  $\mathcal{Y}$  is pro-represented by

$$\Lambda[[\tau]] \times_{\Lambda[[\tau^2]]} \Lambda[[\tau]] \simeq \Lambda[[\tau_1, \tau_2]] / (\tau_1^2 - \tau_2^2),$$

where  $\Lambda$  is smooth over the base scheme  $S$ .

Therefore  $\mathcal{Y}$  is non-normal at  $y$ . However, it may still be that the coarse moduli space of  $\mathcal{Y}$  is normal at the image of  $y$ .  $\square$

**Remark B.2.** So far  $C$  need not have been irreducible: if  $C$  was the union of two smooth curves intersecting at a node, our calculation so far would be the same. However, in this case the coarse moduli space of  $\mathcal{Y}$  would be smooth over  $S$  at  $y$ .

Continuing with our irreducible  $C$ , we claim that the automorphism groups of  $(C, \mathcal{E}_i, b_i)$  act trivially on the base of the deformation functor. Since we assumed  $C$  general we have  $\text{Aut}(C) = 1$ . Thus it remains to calculate  $\text{Aut}(\mathcal{E}_i, b_i)$ .

**Lemma B.3.**  $\text{Aut}(\mathcal{E}_i, b_i) = \{\pm 1\}$ .

*Proof.* Let  $\nu: \tilde{C} \rightarrow C$  be the normalization map,  $p, q \in \tilde{C}$  the pre-images of  $x$  and  $\tilde{N} = \nu^*N$ . Define  $L_i$  to be the line bundle on  $\tilde{C}$  defined as the quotient of  $\nu^*\mathcal{E}_i$  via its torsion. The map  $b_i$  lifts to an isomorphism  $\tilde{b}_i: L_i^{\otimes 2} \simeq \tilde{N}(-p-q)$ . We can recover our root by pushing forward  $(L_i, \tilde{b}_i)$ .

It is easy to see that  $\text{Aut}(\mathcal{E}_i, b_i) \simeq \text{Aut}(L_i, \tilde{b}_i)$ . Since  $L_i$  is a line bundle we may compute the latter and conclude that  $\text{Aut}(\mathcal{E}_i, b_i) = \{\pm 1\}$ , where  $\pm 1$  acts by multiplication.  $\square$

Multiplication by  $\pm 1$  lifts to multiplication by  $\pm 1$  of the root on the universal deformation and therefore does not act on the base.

This concludes our argument that even on the coarse moduli space the image of  $y$  is contained in a non-normal locus. Hence we proved the following.

**Proposition B.4.** *The coarse moduli space of  $\bar{\mathcal{S}}(\mathcal{N}) \times_{\mathcal{M}} \bar{\mathcal{S}}(\mathcal{N})$  is non-normal.*

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